

NORMALITY OF MAXIMAL ORBIT CLOSURES FOR EUCLIDEAN QUIVERS

GRZEGORZ BOBIŃSKI

ABSTRACT. Let Δ be an Euclidean quiver. We prove that the closures of the maximal orbits in the varieties of representations of Δ are normal and Cohen–Macaulay (even complete intersections). Moreover, we give a generalization of this result for the tame concealed-canonical algebras.

INTRODUCTION AND THE MAIN RESULTS

Throughout the paper k is a fixed algebraically closed field. By \mathbb{Z} , \mathbb{N} and \mathbb{N}_+ we denote the sets of the integers, the non-negative integers and the positive integers, respectively. Finally, if $i, j \in \mathbb{Z}$, then $[i, j] := \{l \in \mathbb{Z} \mid i \leq l \leq j\}$ (in particular, $[i, j] = \emptyset$ if $i > j$).

Let A be a finite dimensional k -algebra. Given a non-negative integer d one defines $\text{mod}_A(d)$ as the set of all k -algebra homomorphisms from A to the algebra $\mathbb{M}_{d \times d}(k)$ of $d \times d$ -matrices. This set has a structure of an affine variety and its points represent d -dimensional A -modules. Consequently, we call $\text{mod}_A(d)$ the variety of A -modules of dimension d . The general linear group $\text{GL}(d)$ acts on $\text{mod}_A(d)$ by conjugation: $(g \cdot m)(a) := gm(a)g^{-1}$ for $g \in \text{GL}(d)$, $m \in \text{mod}_A(d)$ and $a \in A$. The orbits with respect to this action are in one-to-one correspondence with the isomorphism classes of the d -dimensional A -modules. Given a d -dimensional A -module M we denote the orbit in $\text{mod}_A(d)$ corresponding to the isomorphism class of M by $\mathcal{O}(M)$ and its Zariski-closure by $\overline{\mathcal{O}(M)}$.

Singularities appearing in the orbit closures $\overline{\mathcal{O}(M)}$ for modules M over an algebra A are an object of intensive studies (see for example [1, 3, 9, 12, 28, 37, 41–44], we also refer to a survey article of Zwara [45]). In particular, Zwara and the author [8] proved that if A is a hereditary algebra of Dynkin type \mathbb{A} or \mathbb{D} , then $\overline{\mathcal{O}(M)}$ is a normal Cohen–Macaulay variety, which has rational singularities if the characteristic of k is 0. Recall, that Gabriel [23] proved that the hereditary algebras of Dynkin type are precisely the hereditary algebras of finite representation type. Thus, it is an interesting question if the orbit closures have good geometric properties for all hereditary algebras of finite representation type. The remaining case of hereditary algebras of type \mathbb{E} is still open, but

there are some partial results in this direction [39]. On the other hand, Zwara [40] exhibited an example of a module over the Kronecker algebra whose orbit closure is neither normal nor Cohen–Macaulay. This example generalizes easily to an arbitrary hereditary algebra of infinite representation type [15]. However, it is still an interesting problem to determine for which classes of modules over hereditary algebras of infinite representation type, the corresponding orbit closures have good properties. In the paper, we study modules M such that $\mathcal{O}(M)$ is maximal, i.e. there is no module N such that $\mathcal{O}(M) \subseteq \overline{\mathcal{O}(N)}$ and $\mathcal{O}(M) \neq \mathcal{O}(N)$.

According to famous Drozd’s Tame and Wild Theorem [17, 21] the finite dimensional algebras of infinite representation type can be divided into two disjoint classes. One class consists of the tame algebras, for which the indecomposable modules of a given dimension form a finite number of one-parameter families. The other class consists of the wild algebras, for which the classification of the indecomposable modules is as complicated as the classification of two non-commuting endomorphisms of a finite dimensional vector space, hence is considered to be hopeless. There are examples showing that varieties of modules over tame algebras have often better properties than those over wild algebras (see for example [6, 16, 35, 36]). Consequently, we concentrate in the paper on the maximal orbits over the tame hereditary algebras. We recall that the tame hereditary algebras are precisely the hereditary algebras of Euclidean type.

The following theorem is the main result of the paper.

Theorem 1. *Let M be a module over a tame hereditary algebra. If $\mathcal{O}(M)$ is maximal, then $\overline{\mathcal{O}(M)}$ is a normal complete intersection (in particular, Cohen–Macaulay).*

It is known (see for example [30, Corollary 3.6]) that $\mathcal{O}(M)$ is maximal for each indecomposable module over a tame hereditary algebra. Consequently, we get the following.

Corollary 2. *If M is an indecomposable module over a tame hereditary algebra, then $\overline{\mathcal{O}(M)}$ is a normal complete intersection (in particular, Cohen–Macaulay).*

Now we present the strategy of the proof of Theorem 1. Let M be a module over a tame hereditary algebra A such that $\mathcal{O}(M)$ is maximal. If $\text{Ext}_A^1(M, M) = 0$, then it is well known that $\overline{\mathcal{O}(M)}$ is smoothly equivalent to an affine space, hence the claim is obvious in this case. Thus we may concentrate on the case $\text{Ext}_A^1(M, M) \neq 0$. It follows from [30, proof of Corollary 3.6] that in this situation M is periodic with respect to the action of the Auslander–Reiten translation τ . Consequently, Theorem 1 follows from the following.

Theorem 3. *Let M be a τ -periodic module over a tame hereditary algebra. If $\mathcal{O}(M)$ is maximal, then $\overline{\mathcal{O}(M)}$ is a complete intersection (in particular, Cohen–Macaulay).*

If A is a tame hereditary algebra, then the τ -periodic A -modules are direct sums of indecomposable modules, which lie in the sincere separating family of tubes in the Auslander–Reiten quiver of A . Existence of such families characterizes the concealed-canonical algebras [27, 34]. Recall [26] that an algebra A is called concealed-canonical if there exists a tilting bundle over a weighted projective line whose endomorphism ring is isomorphic to A . Thus it is natural to try to generalize Theorem 3 to the case of tame concealed-canonical algebras. Before we formulate this generalization, we present necessary definitions.

Let A be a tame concealed-canonical algebra. For an A -module M we denote by $\mathbf{dim} M$ its dimension vector, i.e. the sequence indexed by the isomorphism classes of the simple A -modules, which counts the multiplicities of the composition factors in the Jordan–Hölder filtration of M . In general, a sequence of non-negative integers indexed by the isomorphism classes of the simple A -modules is called a dimension vector. We call a dimension vector \mathbf{d} singular if $\langle \mathbf{d}, \mathbf{d} \rangle_A = 0$ and there exists a dimension vector \mathbf{x} such that $\mathbf{x} \leq \mathbf{d}$, $\langle \mathbf{x}, \mathbf{x} \rangle_A = 0$ and $|\langle \mathbf{x}, \mathbf{d} \rangle_A| = 2$, where $\langle -, - \rangle_A$ denotes the corresponding homological bilinear form (see Section 1). In Proposition 2.3 we describe the tame concealed-canonical algebras for which there exist singular dimension vectors. In particular, this description implies that singular dimension vectors do not exist for the tame hereditary algebras.

We have the following generalization of Theorem 3.

Theorem 4. *Let M be a τ -periodic module over a tame concealed-canonical algebra such that $\mathcal{O}(M)$ is maximal. Then $\overline{\mathcal{O}(M)}$ is a complete intersection (in particular, Cohen–Macaulay). Moreover, $\overline{\mathcal{O}(M)}$ is not normal if and only if $\mathbf{dim} M$ is singular and $\tau M \simeq M$.*

In the paper we concentrate on the proof of Theorem 4. Instead of using the framework of modules over algebras and the corresponding varieties, we use the framework of representations of quivers (and the corresponding varieties). Gabriel’s Theorem [23] says that we may do this replacement on the level of modules and representations, while a result of Bongartz [11] justifies this passage on the level of varieties. For the background on the representation theory we refer to [2, 32, 33].

The paper is organized as follows. In Section 1 we recall basic information about quivers and their representations. Next, in Section 2 we gather facts about the categories of modules over the tame concealed-canonical algebras. In Section 3 we introduce varieties of representations of quivers, while in Section 4 we review facts on semi-invariants with particular emphasis on the case of tame concealed-canonical algebras. Next, in Section 5 we present a series of facts, which we later

use in Sections 6 and 7 to study orbit closures for the non-singular and singular dimension vectors, respectively. Moreover, in Section 7 we make a remark about relationship between the degenerations and the hom-order for the tame concealed-canonical algebras. Finally, in Section 8 we give the proof of Theorem 4.

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1. QUIVERS AND THEIR REPRESENTATIONS

By a quiver Δ we mean a finite set Δ_0 (called the set of vertices of Δ) together with a finite set Δ_1 (called the set of arrows of Δ) and two maps $s, t : \Delta_1 \rightarrow \Delta_0$, which assign to each arrow α its starting vertex $s\alpha$ and terminating vertex $t\alpha$, respectively. By a path of length $n \in \mathbb{N}_+$ in a quiver Δ we mean a sequence $\sigma = (\alpha_1, \dots, \alpha_n)$ of arrows such that $s\alpha_i = t\alpha_{i+1}$ for each $i \in [1, n-1]$. In particular, we treat every arrow in Δ as a path of length 1. In the above situation we put $\ell\sigma := n$, $s\sigma := s\alpha_n$ and $t\sigma := t\alpha_1$. Moreover, for each vertex x we have a trivial path $\mathbf{1}_x$ at x such that $\ell\mathbf{1}_x := 0$ and $s\mathbf{1}_x := x =: t\mathbf{1}_x$. A subquiver Δ' of a quiver Δ is called convex if $\alpha_i \in \Delta'_1$ for each $i \in [1, n]$, provided $(\alpha_1, \dots, \alpha_n)$ is a path in Δ such that $t\alpha_1, s\alpha_n \in \Delta'_0$.

For the rest of the paper we assume that the considered quivers do not have oriented cycles, where by an oriented cycle we mean a path σ of positive length such that $s\sigma = t\sigma$.

Let Δ be a quiver. We define its path category $k\Delta$ to be the category whose objects are the vertices of Δ and, for $x, y \in \Delta_0$, the morphisms from x to y are the formal k -linear combinations of paths starting at x and terminating at y . For $x, y \in \Delta_0$ we denote by $k\Delta(x, y)$ the space of the morphisms from x to y in $k\Delta$. If $\omega \in k\Delta(x, y)$ for $x, y \in \Delta_0$, then we write $s\omega := x$ and $t\omega := y$. By a representation of Δ we mean a functor from $k\Delta$ to the category $\text{mod } k$ of finite dimensional vector spaces. We denote the category of the representations of Δ by $\text{rep } \Delta$. Observe that every representation of Δ is uniquely determined by its values on the vertices and the arrows. Given a representation M of Δ we denote by $\mathbf{dim} M$ its dimension vector defined by $(\mathbf{dim} M)(x) := \dim_k M(x)$ for $x \in \Delta_0$. Observe the $\mathbf{dim} M \in \mathbb{N}^{\Delta_0}$ for each representation M of Δ . We call the elements of \mathbb{N}^{Δ_0} dimension vectors. A dimension vector \mathbf{d} is called sincere if $\mathbf{d}(x) \neq 0$ for each $x \in \Delta_0$.

By a relation in a quiver Δ we mean a k -linear combination of paths of lengths at least 2 having a common starting vertex and a common terminating vertex. Note that each relation in a quiver Δ is a morphism in $k\Delta$. A set R of relations in a quiver Δ is called minimal if $\langle R \setminus \{\rho\} \rangle \neq \langle R \rangle$ for each $\rho \in R$, where for a set X of morphisms in Δ we denote by $\langle X \rangle$ the ideal in $k\Delta$ generated by X . Observe that each minimal set of relations is finite. By a bound quiver Δ we mean a quiver Δ together with a minimal set R of relations. Given a bound quiver Δ

we denote by $k\Delta$ its path category, i.e. $k\Delta := k\Delta/\langle R \rangle$. Moreover, for $x, y \in \Delta_0$ we denote by $k\Delta(x, y)$ the space of the morphisms from x to y in $k\Delta$. By a representation of a bound quiver Δ we mean a functor from $k\Delta$ to $\text{mod } k$. In other words, a representation of Δ is a representation M of Δ such that $M(\rho) = 0$ for each $\rho \in R$. We denote the category of the representations of a bound quiver Δ by $\text{rep } \Delta$. Moreover, we denote by $\text{ind } \Delta$ the full subcategory of $\text{rep } \Delta$ consisting of the indecomposable representations. It is known that $\text{rep } \Delta$ is an abelian Krull–Schmidt category. A bound quiver Δ' is called a convex subquiver of a bound quiver Δ if Δ' is a convex subquiver of Δ and $R' = R \cap k\Delta'$. If Δ' is a convex subquiver of a bound quiver Δ , then $\text{rep } \Delta'$ can be naturally identified with an exact subcategory of $\text{rep } \Delta$, where by an exact subcategory of $\text{rep } \Delta$ we mean a full subcategory \mathcal{X} of $\text{rep } \Delta$ such that \mathcal{X} is an abelian category and the inclusion functor $\mathcal{X} \hookrightarrow \text{rep } \Delta$ is exact. In particular, if Δ' is a convex subcategory of a tame bound quiver Δ , then Δ' is either tame or representation-finite (we say that a bound quiver Δ is tame/representation-finite if $\text{rep } \Delta$ is of tame/finite representation type, respectively).

Let Δ be a bound quiver. For each vertex x of Δ we denote by S_x the simple representation at x , i.e. $S_x(x) := k$, $S_x(y) := 0$ for $y \in \Delta_0 \setminus \{x\}$, and $S_x(\alpha) := 0$ for $\alpha \in \Delta_1$. More generally, if \mathbf{d} is a dimension vector, then we put $S^{\mathbf{d}} := \bigoplus_{x \in \Delta_0} S_x^{\mathbf{d}(x)}$. Next, for each vertex x we denote by P_x the projective representation at x defined in the following way: $P_x(y) := k\Delta(x, y)$ for $y \in \Delta_0$ and $P_x(\omega)$ is the composition (on the left) with ω for a morphism ω in $k\Delta$. If M is a representation of Δ and $x \in \Delta_0$, then the map

$$\text{Hom}_{\Delta}(P_x, M) \rightarrow M(x), \quad f \mapsto f(\mathbf{1}_x),$$

is an isomorphism. In particular, this implies that

$$\text{Hom}_{\Delta}(P_x, P_y) \simeq k\Delta(y, x)$$

for any $x, y \in \Delta_0$. If $\omega \in k\Delta(y, x)$, we denote the corresponding map $P_x \rightarrow P_y$ by P_{ω} . Observe that P_{ω} is the composition (on the right) with ω . Moreover, if M is a representation of Δ , then, under the above isomorphisms, $\text{Hom}_{\Delta}(P_{\omega}, M)$ equals $M(\omega)$.

Let Δ be a bound quiver. If $P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0$ is a (fixed) minimal projective presentation of a representation M of Δ , then we put

$$\tau M := \text{Hom}_k(\text{Coker Hom}_{\Delta}(f, \bigoplus_{x \in \Delta_0} P_x), k).$$

We define $\tau^- M$ dually. Note that $\tau M = 0$ ($\tau^- M = 0$) if and only if M is projective (injective, respectively). Moreover, $\tau^- \tau X \simeq X$ ($\tau \tau^- X \simeq X$) for each indecomposable representation X of Δ , which is not projective (injective, respectively). We say that a representation M of Δ is periodic if there exists $n \in \mathbb{N}_+$ such that $\tau^n M \simeq M$. We

have a celebrated Auslander–Reiten formula [2, Section IV.2], which implies that

$$\dim_k \operatorname{Ext}_{\Delta}^1(M, N) = \dim_k \operatorname{Hom}_{\Delta}(N, \tau M)$$

for any representations M and N of Δ such that $\operatorname{pdim}_{\Delta} M \leq 1$. Dually, if M and N are representations of Δ and $\operatorname{idim}_{\Delta} N \leq 1$, then

$$\dim_k \operatorname{Ext}_{\Delta}^1(M, N) = \dim_k \operatorname{Hom}_{\Delta}(\tau^{-} N, M).$$

Let Δ be a bound quiver. We define the corresponding Tits forms $\langle -, - \rangle_{\Delta} : \mathbb{Z}^{\Delta_0} \times \mathbb{Z}^{\Delta_0} \rightarrow \mathbb{Z}$ and $q_{\Delta} : \mathbb{Z}^{\Delta_0} \rightarrow \mathbb{Z}$ by

$$\langle \mathbf{d}', \mathbf{d}'' \rangle_{\Delta} := \sum_{x \in \Delta_0} \mathbf{d}'(x) \mathbf{d}''(x) - \sum_{\alpha \in \Delta_1} \mathbf{d}'(s\alpha) \mathbf{d}''(t\alpha) + \sum_{\rho \in R} \mathbf{d}'(s\rho) \mathbf{d}''(t\rho)$$

for $\mathbf{d}', \mathbf{d}'' \in \mathbb{Z}^{\Delta_0}$, and $q_{\Delta}(\mathbf{d}) := \langle \mathbf{d}, \mathbf{d} \rangle_{\Delta}$ for $\mathbf{d} \in \mathbb{Z}^{\Delta_0}$. Bongartz [10, Proposition 2.2] proved that

$$\begin{aligned} \langle \dim M, \dim N \rangle_{\Delta} \\ = \dim_k \operatorname{Hom}_{\Delta}(M, N) - \dim_k \operatorname{Ext}_{\Delta}^1(M, N) + \dim_k \operatorname{Ext}_{\Delta}^2(M, N) \end{aligned}$$

for any $M, N \in \operatorname{rep} \Delta$, provided $\operatorname{gl. dim} \Delta \leq 2$.

2. SEPARATING EXACT SUBCATEGORIES

In this section we present facts about sincere separating exact subcategories, which we use in our considerations. For the proofs we refer to [27, 31].

Let Δ be a bound quiver and \mathcal{X} a full subcategory of $\operatorname{ind} \Delta$. We denote by $\operatorname{add} \mathcal{X}$ the full subcategory of $\operatorname{rep} \Delta$ formed by the direct sums of representations from \mathcal{X} . We say that \mathcal{X} is an exact subcategory of $\operatorname{ind} \Delta$ if $\operatorname{add} \mathcal{X}$ is an exact subcategory of $\operatorname{rep} \Delta$. We put

$$\mathcal{X}_+ := \{X \in \operatorname{ind} \Delta : \operatorname{Hom}_{\Delta}(\mathcal{X}, X) = 0\}$$

and

$$\mathcal{X}_- := \{X \in \operatorname{ind} \Delta : \operatorname{Hom}_{\Delta}(X, \mathcal{X}) = 0\}.$$

Let Δ be a bound quiver. Following [27] we say that \mathcal{R} is a sincere separating exact subcategory of $\operatorname{ind} \Delta$ provided the following conditions are satisfied:

- (1) \mathcal{R} is an exact subcategory of $\operatorname{ind} \Delta$ consisting of periodic representations.
- (2) $\operatorname{ind} \Delta = \mathcal{R}_+ \cup \mathcal{R} \cup \mathcal{R}_-$.
- (3) $\operatorname{Hom}_{\Delta}(X, \mathcal{R}) \neq 0$ for each $X \in \mathcal{R}_+$ and $\operatorname{Hom}_{\Delta}(\mathcal{R}, X) \neq 0$ for each $X \in \mathcal{R}_-$.
- (4) $P \in \mathcal{R}_+$ for each indecomposable projective representation P of Δ and $I \in \mathcal{R}_-$ for each indecomposable injective representation I of Δ .

Lenzing and de la Peña [27] proved that there exists a sincere separating exact subcategory \mathcal{R} of $\text{rep } \Delta$ if and only if Δ is concealed-canonical, i.e. $\text{rep } \Delta$ is equivalent to the category of modules over a concealed-canonical algebra. In particular, if this the case, then $\text{gl. dim } \Delta \leq 2$.

For the rest of the section we fix a bound quiver Δ and a sincere separating exact subcategory \mathcal{R} of $\text{ind } \Delta$. Moreover, we put $\mathcal{P} := \mathcal{R}_+$ and $\mathcal{Q} := \mathcal{R}_-$. Finally, we denote by \mathbf{P} , \mathbf{R} and \mathbf{Q} the dimension vectors of the representations from $\text{add } \mathcal{P}$, $\text{add } \mathcal{R}$ and $\text{add } \mathcal{Q}$, respectively.

It is known that $\text{pdim}_\Delta P \leq 1$ for each $P \in \mathcal{P}$ and $\text{idim}_\Delta Q \leq 1$ for each $Q \in \mathcal{Q}$. Next, $\text{pdim}_\Delta R = 1$ and $\text{idim}_\Delta R = 1$ for each $R \in \mathcal{R}$. Moreover, $\text{Hom}_\Delta(\mathcal{Q}, \mathcal{P}) = 0$. Since the categories \mathcal{P} and \mathcal{Q} are closed under the actions of τ and τ^- , using the Auslander–Reiten formulas we also obtain that $\text{Ext}_\Delta^1(\mathcal{P}, \mathcal{R} \cup \mathcal{Q}) = 0 = \text{Ext}_\Delta^1(\mathcal{P} \cup \mathcal{R}, \mathcal{Q})$. The above properties imply that $\langle \mathbf{d}', \mathbf{d}'' \rangle_\Delta \geq 0$ if either $\mathbf{d}' \in \mathbf{P}$ and $\mathbf{d}'' \in \mathbf{R} + \mathbf{Q}$ or $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$. Similarly, $\langle \mathbf{d}'', \mathbf{d}' \rangle_\Delta \leq 0$ if either $\mathbf{d}' \in \mathbf{P}$ and $\mathbf{d}'' \in \mathbf{R}$ or $\mathbf{d}' \in \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$.

We have $\mathcal{R} = \coprod_{\lambda \in \mathbb{X}} \mathcal{R}_\lambda$ for some infinite set \mathbb{X} and connected uniserial categories \mathcal{R}_λ , $\lambda \in \mathbb{X}$. For $\lambda \in \mathbb{X}$ we denote by r_λ the number of the pairwise non-isomorphic simple objects in $\text{add } \mathcal{R}_\lambda$. Then $r_\lambda < \infty$. Let $\mathbb{X}_0 := \{\lambda \in \mathbb{X} : r_\lambda > 1\}$. Then $|\mathbb{X}_0| < \infty$ and we call the sequence $(r_\lambda)_{\lambda \in \mathbb{X}_0}$ the type of Δ (this definition does not depend on the choice of a sincere separating exact subcategory of $\text{ind } \Delta$). It is known that Δ is tame if and only if $\sum_{\lambda \in \mathbb{X}_0} \frac{1}{r_\lambda} \geq |\mathbb{X}_0| - 2$, where by definition the empty sum equals 0. Observe that this implies that $|\mathbb{X}_0| \leq 4$ provided Δ is tame. Moreover, if Δ is tame and $|\mathbb{X}_0| = 4$, then Δ is of type $(2, 2, 2, 2)$.

Fix $\lambda \in \mathbb{X}$. If $R_{\lambda,0}, \dots, R_{\lambda,r_\lambda-1}$ are chosen representatives of the isomorphism classes of the simple objects in $\text{add } \mathcal{R}_\lambda$, then we may assume that $\tau R_{\lambda,i} = R_{\lambda,i-1}$ for each $i \in [0, r_\lambda - 1]$, where we put $R_{\lambda,i} := R_{\lambda,i \bmod r_\lambda}$ for $i \in \mathbb{Z}$. For $i \in \mathbb{Z}$ and $n \in \mathbb{N}_+$ there exists a unique (up to isomorphism) representation in \mathcal{R}_λ , whose top and length in $\text{add } \mathcal{R}_\lambda$ are $R_{\lambda,i}$ and n , respectively. We fix such representation and denote it by $R_{\lambda,i}^{(n)}$, and its dimension vector by $\mathbf{e}_{\lambda,i}^n$. Then the composition factors of $R_{\lambda,i}^{(n)}$ are (starting from the top): $R_{\lambda,i}, R_{\lambda,i-1}, \dots, R_{\lambda,i-(n-1)}$. Consequently, $\mathbf{e}_{\lambda,i}^n = \sum_{j \in [i-n+1, i]} \mathbf{e}_{\lambda,j}$, where $\mathbf{e}_{\lambda,j} := \mathbf{dim } R_{\lambda,j}$ for $j \in \mathbb{Z}$. Moreover, if $i \in \mathbb{Z}$ and $m, n \in \mathbb{N}_+$, then we have an exact sequence $0 \rightarrow R_{\lambda,i-n}^{(m)} \rightarrow R_{\lambda,i}^{(m+n)} \rightarrow R_{\lambda,i}^{(n)} \rightarrow 0$. Obviously, for each $R \in \mathcal{R}_\lambda$ there exist $i \in \mathbb{Z}$ and $n \in \mathbb{N}_+$ such that $R \simeq R_{\lambda,i}^{(n)}$. Moreover, it is known that the vectors $\mathbf{e}_{\lambda,0}, \dots, \mathbf{e}_{\lambda,r_\lambda-1}$ are linearly independent. Consequently, if $R \in \text{add } \mathcal{R}_\lambda$, then there exist uniquely determined $q_0^R, \dots, q_{r_\lambda-1}^R \in \mathbb{N}$ such that $\mathbf{dim } R = \sum_{i \in [0, r_\lambda-1]} q_i^R \mathbf{e}_{\lambda,i}$. Observe that the numbers $q_{\lambda,0}^R, \dots, q_{\lambda,r_\lambda-1}^R$ count the multiplicities in which the modules $R_{\lambda,0}, \dots, R_{\lambda,r_\lambda-1}$ appear as composition factors in the Jordan–Hölder filtration of R in the category $\text{add } \mathcal{R}_\lambda$.

Let $R = \bigoplus_{\lambda \in \mathbb{X}} R_\lambda$ for $R_\lambda \in \text{add } \mathcal{R}_\lambda$, $\lambda \in \mathbb{X}$. Then we put $q_{\lambda,i}^R := q_i^{R_\lambda}$ for $\lambda \in \mathbb{X}$ and $i \in [0, r_\lambda - 1]$. Next, we put $p_\lambda^R := \min\{q_{\lambda,i}^R : i \in [0, r_\lambda - 1]\}$ for $\lambda \in \mathbb{X}$, and $p_{\lambda,i}^R := q_{\lambda,i}^R - p_\lambda^R$ for $\lambda \in \mathbb{X}$ and $i \in [0, r_\lambda - 1]$. Then

$$\dim R = \sum_{\lambda \in \mathbb{X}} p_\lambda^R \mathbf{h}_\lambda + \sum_{\lambda \in \mathbb{X}} \sum_{i \in [0, r_\lambda - 1]} p_{\lambda,i}^R \mathbf{e}_{\lambda,i},$$

where $\mathbf{h}_\lambda := \sum_{i \in [0, r_\lambda - 1]} \mathbf{e}_{\lambda,i}$ for $\lambda \in \mathbb{X}$. It is known that $\mathbf{h}_\lambda = \mathbf{h}_\mu$ for any $\lambda, \mu \in \mathbb{X}$. We denote this common value by \mathbf{h} . Then

$$\dim R = p^R \mathbf{h} + \sum_{\lambda \in \mathbb{X}} \sum_{i \in [0, r_\lambda - 1]} p_{\lambda,i}^R \mathbf{e}_{\lambda,i},$$

where $p^R := \sum_{\lambda \in \mathbb{X}} p_\lambda^R$. It is known that $p^R = p^{R'}$ and $p_{\lambda,i}^R = p_{\lambda,i}^{R'}$ for any $\lambda \in \mathbb{X}$ and $i \in [0, r_\lambda - 1]$, if $R, R' \in \text{add } \mathcal{R}$ and $\dim R = \dim R'$. Consequently, for each $\mathbf{d} \in \mathbf{R}$ there exist uniquely determined $p^{\mathbf{d}} \in \mathbb{N}$ and $p_{\lambda,i}^{\mathbf{d}} \in \mathbb{N}$, $\lambda \in \mathbb{X}$, $i \in [0, r_\lambda - 1]$, such that for each $\lambda \in \mathbb{X}$ there exists $i \in [0, r_\lambda - 1]$ with $p_{\lambda,i}^{\mathbf{d}} = 0$ and

$$\mathbf{d} = p^{\mathbf{d}} \mathbf{h} + \sum_{\lambda \in \mathbb{X}} \sum_{i \in [0, r_\lambda - 1]} p_{\lambda,i}^{\mathbf{d}} \mathbf{e}_{\lambda,i}.$$

Let $\lambda, \mu \in \mathbb{X}$, $i, j \in \mathbb{Z}$, and $m, n \in \mathbb{N}_+$. Then

$$\dim_k \text{Hom}_\Delta(R_{\lambda,i}^{(n)}, R_{\mu,j}^{(m)}) = \min\{q_{\lambda,i \bmod r_\lambda}^{R_{\mu,j}^{(m)}}, q_{\mu,(j-m+1) \bmod r_\lambda}^{R_{\lambda,i}^{(n)}}\}.$$

In particular, if $\lambda \in \mathbb{X}$, $i \in [0, r_\lambda - 1]$, $n \in \mathbb{N}_+$, $R \in \text{add } \mathcal{R}$ and $\text{Hom}_\Delta(R_{\lambda,i}^{(n)}, R) \neq 0$, then $q_{\lambda,i}^R \neq 0$. Moreover, the above formula together with the Auslander–Reiten formula imply that

$$\langle \mathbf{e}_{i,\lambda}^n, \mathbf{d} \rangle_\Delta = p_{\lambda,i \bmod r_\lambda}^{\mathbf{d}} - p_{\lambda,(i-n) \bmod r_\lambda}^{\mathbf{d}}$$

and

$$\langle \mathbf{d}, \mathbf{e}_{i,\lambda}^n \rangle_\Delta = p_{\lambda,(i-n+1) \bmod r_\lambda}^{\mathbf{d}} - p_{\lambda,(i+1) \bmod r_\lambda}^{\mathbf{d}}$$

for any $\lambda \in \mathbb{X}$, $i \in \mathbb{Z}$, $n \in \mathbb{N}_+$, and $\mathbf{d} \in \mathbf{R}$. Consequently, $\langle \mathbf{h}, \mathbf{d} \rangle_\Delta = 0 = \langle \mathbf{d}, \mathbf{h} \rangle_\Delta$ for each $\mathbf{d} \in \mathbf{R}$. In particular, $q_\Delta(\mathbf{h}) = 0$. On the other hand, if $\mathbf{d} \in \mathbf{R}$, then $q_\Delta(\mathbf{d}) = 0$ if and only if $\mathbf{d} = p^{\mathbf{d}} \mathbf{h}$. One also shows that \mathbf{h} is indivisible.

We also need some other properties of the Tits form, which we list now.

Proposition 2.1. *Assume that Δ is tame. Then the following hold.*

- (1) $q_\Delta(\mathbf{d}) \geq 0$ for each dimension vector \mathbf{d} .
- (2) If $q_\Delta(\mathbf{d}) = 0$ for a dimension vector \mathbf{d} , then $\mathbf{d} \in \mathbf{P} \cup \mathbf{R} \cup \mathbf{Q}$ and $\langle \mathbf{d}, \mathbf{d}_0 \rangle_\Delta + \langle \mathbf{d}_0, \mathbf{d} \rangle_\Delta = 0$ for each dimension vector \mathbf{d}_0 .
- (3) If $\mathbf{d} \in \mathbf{P} \cup \mathbf{Q}$ is non-zero, then $\langle \mathbf{d}, \mathbf{h} \rangle_\Delta \neq 0$.

- (4) If $\mathbf{d} \in \mathbf{P} \cup \mathbf{Q}$ is non-zero and $q_{\Delta}(\mathbf{d}) = 0$, then $\langle \mathbf{d}, \mathbf{d}_0 \rangle_{\Delta} \neq 0$ for each non-zero vector $\mathbf{d}_0 \in \mathbf{R}$. In particular,

$$|\langle \mathbf{d}, \mathbf{h} \rangle_{\Delta}| \geq \max\{r_{\lambda} : \lambda \in \mathbb{X}\}.$$

- (5) If there exists non-zero $\mathbf{d} \in \mathbf{P} \cup \mathbf{Q}$ such that $q_{\Delta}(\mathbf{d}) = 0$, then $\sum_{\lambda \in \mathbb{X}_0} \frac{1}{r_{\lambda}} = |\mathbb{X}_0| - 2$. In particular, if this is the case, then $\max\{r_{\lambda} : \lambda \in \mathbb{X}\} \geq 2$ and $\max\{r_{\lambda} : \lambda \in \mathbb{X}\} = 2$ if and only if Δ is of type $(2, 2, 2, 2)$.

As a consequence we obtain the following.

Corollary 2.2. *Let $\mathbf{d} \in \mathbf{R}$, $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$. If $p^{\mathbf{d}} > 0$, $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$ and $\mathbf{d}'' \neq 0$, then $\langle \mathbf{d}'', \mathbf{d}' \rangle_{\Delta} \leq -p^{\mathbf{d}} - 1$. Moreover, $\langle \mathbf{d}'', \mathbf{d}' \rangle_{\Delta} = -p^{\mathbf{d}} - 1$ if and only if one of the following conditions is satisfied:*

- (1) $q_{\Delta}(\mathbf{d}'') = 1$ and $\langle \mathbf{d}'', \mathbf{d} \rangle_{\Delta} = -p^{\mathbf{d}}$ (in particular, $\langle \mathbf{d}'', \mathbf{h} \rangle_{\Delta} = -1$),
or
- (2) $q_{\Delta}(\mathbf{d}'') = 0$ and $\langle \mathbf{d}'', \mathbf{d} \rangle_{\Delta} = -2$.

Proof. Put $\mathbf{d}_0 := \mathbf{d} - p^{\mathbf{d}}\mathbf{h}$. Then $\mathbf{d}_0 \in \mathbf{R}$. We have

$$\langle \mathbf{d}'', \mathbf{d}' \rangle_{\Delta} = \langle \mathbf{d}'', \mathbf{d} - \mathbf{d}'' \rangle_{\Delta} = -q_{\Delta}(\mathbf{d}'') + p^{\mathbf{d}}\langle \mathbf{d}'', \mathbf{h} \rangle_{\Delta} + \langle \mathbf{d}'', \mathbf{d}_0 \rangle_{\Delta}.$$

Now $\langle \mathbf{d}'', \mathbf{d}_0 \rangle_{\Delta} \leq 0$. Moreover, $\langle \mathbf{d}'', \mathbf{h} \rangle_{\Delta} \leq -1$ and $q_{\Delta}(\mathbf{d}'') \geq 0$. Finally, if $q_{\Delta}(\mathbf{d}'') = 0$, then $\langle \mathbf{d}'', \mathbf{h} \rangle_{\Delta} \leq -2$, hence the inequality follows.

These considerations also imply that $\langle \mathbf{d}'', \mathbf{d}' \rangle_{\Delta} = -p^{\mathbf{d}} - 1$ if and only if one of the following conditions is satisfied:

- (1) $q_{\Delta}(\mathbf{d}'') = 1$, $\langle \mathbf{d}'', \mathbf{h} \rangle_{\Delta} = -1$ and $\langle \mathbf{d}'', \mathbf{d}_0 \rangle_{\Delta} = 0$, or
- (2) $q_{\Delta}(\mathbf{d}'') = 0$, $p^{\mathbf{d}} = 1$, $\langle \mathbf{d}'', \mathbf{h} \rangle_{\Delta} = -2$ and $\langle \mathbf{d}'', \mathbf{d}_0 \rangle_{\Delta} = 0$.

These conditions immediately lead to (and follows from) the conditions given in the corollary. \square

We call a dimension vector $\mathbf{d} \in \mathbf{R}$ singular if $p^{\mathbf{d}} > 0$ and there exists a dimension vector \mathbf{x} such that $\mathbf{x} \leq \mathbf{d}$, $q_{\Delta}(\mathbf{x}) = 0$ and $|\langle \mathbf{x}, \mathbf{d} \rangle_{\Delta}| = 2$. It follows from the below proposition that this definition coincides the the definition given in the introduction.

Proposition 2.3. *Let $\mathbf{d} \in \mathbf{R}$ be such that $p^{\mathbf{d}} > 0$.*

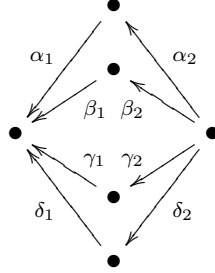
- (1) *If \mathbf{d} is singular, then $\mathbf{d} = \mathbf{h}$ and Δ is of type $(2, 2, 2, 2)$.*
- (2) *There exist $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$ such that $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$, $q_{\Delta}(\mathbf{d}'') = 0$ and $\langle \mathbf{d}'', \mathbf{d} \rangle_{\Delta} = -2$, if and only if \mathbf{d} is singular.*

Proof. (1) Fix a dimension vector \mathbf{x} such that $\mathbf{x} \leq \mathbf{d}$, $q_{\Delta}(\mathbf{x}) = 0$ and $|\langle \mathbf{x}, \mathbf{d} \rangle_{\Delta}| = 2$. Proposition 2.1(2) implies that $\mathbf{x} \in \mathbf{P} \cup \mathbf{R} \cup \mathbf{Q}$. Since $\langle \mathbf{x}, \mathbf{d} \rangle_{\Delta} \neq 0$, $\mathbf{x} \notin \mathbf{R}$. In particular, \mathbf{x} is non-zero. By symmetry, we may assume $\mathbf{x} \in \mathbf{P}$. If $\mathbf{d}_0 := \mathbf{d} - p^{\mathbf{d}}\mathbf{h}$, then $2 = p^{\mathbf{d}}\langle \mathbf{x}, \mathbf{h} \rangle_{\Delta} + \langle \mathbf{x}, \mathbf{d}_0 \rangle_{\Delta}$. Using Proposition 2.1(4) and (5) we obtain that $p^{\mathbf{d}} = 1$ and $\mathbf{d}_0 = 0$, i.e. $\mathbf{d} = \mathbf{h}$. Moreover, Δ must be of type $(2, 2, 2, 2)$ by Proposition 2.1(5).

(2) One implication is obvious. Now assume there exists a dimension vector \mathbf{x} such that $\mathbf{x} \leq \mathbf{d}$, $q_{\Delta}(\mathbf{x}) = 0$ and $|\langle \mathbf{x}, \mathbf{d} \rangle_{\Delta}| = 2$. From (1) we

know that $\mathbf{d} = \mathbf{h}$. Easy calculations show that $\langle \mathbf{h}, \mathbf{h} - \mathbf{x} \rangle_{\Delta} = -\langle \mathbf{h}, \mathbf{x} \rangle_{\Delta}$ and $q_{\Delta}(\mathbf{h} - \mathbf{x}) = 0$. Thus, Proposition 2.1(2) implies that, up to symmetry, $\mathbf{x} \in \mathbf{P}$ and $\mathbf{h} - \mathbf{x} \in \mathbf{Q}$, and the claim follows. \square

We finish this section with an example showing that singular dimension vectors exist. Fix $\lambda \in k \setminus \{0, 1\}$. Let Δ be the quiver



and $R := \{\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2, \alpha_1\alpha_2 + \beta_1\beta_2 + \lambda\delta_1\delta_2\}$. Then Δ is a concealed-canonical algebra of type $(2, 2, 2, 2)$ (in fact, it is one of Ringel's canonical algebras [30]). Moreover, the vector $\begin{smallmatrix} 2 \\ 3 \\ 2 \\ 2 \\ 1 \end{smallmatrix}$ is singular

– the corresponding vector \mathbf{x} can be taken to be $\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$ (the other choice is $\begin{smallmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$).

3. VARIETIES OF REPRESENTATIONS

First we recall some facts from algebraic geometry. Let \mathcal{X} be a closed subvariety of an affine space \mathbb{A}^n , $n \in \mathbb{N}$. We say that \mathcal{X} is a complete intersection if there exist polynomials $f_1, \dots, f_m \in k[\mathbb{A}^n]$ such that $\dim \mathcal{X} = n - m$ and

$$\{f \in k[\mathbb{A}^n] : f(x) = 0 \text{ for each } x \in \mathcal{X}\} = (f_1, \dots, f_m).$$

For $x \in \mathcal{X}$ we denote by $T_x\mathcal{X}$ the tangent space to \mathcal{X} at x . We will use the following consequences of Serre's criterion (see for example [22, Theorem 18.15]).

Proposition 3.1. *Let \mathcal{X} be a complete intersection.*

- (1) *Let $\mathcal{U} := \{x \in \mathcal{X} : \dim_k T_x\mathcal{X} = \dim \mathcal{X}\}$. Then \mathcal{X} is normal if and only if $\dim(\mathcal{X} \setminus \mathcal{U}) < \dim \mathcal{X} - 1$.*
- (2) *Let $f_1, \dots, f_m \in k[\mathcal{X}]$,*

$$\mathcal{Y} := \{x \in \mathcal{X} : f_i(x) = 0 \text{ for each } i \in [1, m]\}$$

and

$$\mathcal{U} := \{x \in \mathcal{Y} : \partial f_1(x), \dots, \partial f_m(x) \text{ are linearly independent}\}.$$

If $\mathcal{U} \cap \mathcal{C} \neq \emptyset$ for each irreducible component \mathcal{C} of \mathcal{Y} , then

$$\{f \in k[\mathcal{X}] : f(x) = 0 \text{ for each } x \in \mathcal{Y}\} = (f_1, \dots, f_m).$$

In particular, \mathcal{Y} is a complete intersection of dimension $\dim \mathcal{X} - m$. \square

Let Δ be a bound quiver and \mathbf{d} a dimension vector. By $\text{rep}_\Delta(\mathbf{d})$ we denote the set of the representations M of Δ such that $M(x) = k^{\mathbf{d}(x)}$ for each $x \in \Delta_0$. We may identify $\text{rep}_\Delta(\mathbf{d})$ with the affine space $\prod_{\alpha \in \Delta_1} \text{M}_{\mathbf{d}(t\alpha) \times \mathbf{d}(s\alpha)}(k)$. The group $\text{GL}(\mathbf{d}) := \prod_{x \in \Delta_0} \text{GL}(\mathbf{d}(x))$ acts on $\text{rep}_\Delta(\mathbf{d})$ by conjugation: $(g \cdot M)(\alpha) := g(t\alpha)M(\alpha)g(s\alpha)^{-1}$ for $g \in \text{GL}(\mathbf{d})$, $M \in \text{rep}_\Delta(\mathbf{d})$ and $\alpha \in \Delta_1$. Under this action the $\text{GL}(\mathbf{d})$ -orbits in $\text{rep}_\Delta(\mathbf{d})$ correspond to the isomorphism classes of the representations of Δ with dimension vector \mathbf{d} . We denote the $\text{GL}(\mathbf{d})$ -orbit of a representation $M \in \text{rep}_\Delta(\mathbf{d})$ by $\mathcal{O}(M)$.

Now let $\mathbf{\Delta}$ be a bound quiver and \mathbf{d} a dimension vector. By $\text{rep}_{\mathbf{\Delta}}(\mathbf{d})$ we denote the intersection of $\text{rep}_\Delta(\mathbf{d})$ with $\text{rep } \mathbf{\Delta}$. Then $\text{rep}_{\mathbf{\Delta}}(\mathbf{d})$ is a closed $\text{GL}(\mathbf{d})$ -invariant subset of $\text{rep}_\Delta(\mathbf{d})$ and we call it the variety of representations of $\mathbf{\Delta}$ of dimension vector \mathbf{d} . If $M, N \in \text{rep}_{\mathbf{\Delta}}(\mathbf{d})$ and there exists an exact sequence $0 \rightarrow N' \rightarrow M \rightarrow N'' \rightarrow 0$ such that $N \simeq N' \oplus N''$, then $N \in \overline{\mathcal{O}(M)}$. In particular, $S^{\mathbf{d}} \in \overline{\mathcal{O}(M)}$ for each $M \in \text{rep}_{\mathbf{\Delta}}(\mathbf{d})$. If \mathcal{V} is a $\text{GL}(\mathbf{d})$ -invariant subset of $\text{rep}_{\mathbf{\Delta}}(\mathbf{d})$ and $M \in \mathcal{V}$, then we say that the orbit $\mathcal{O}(M)$ is maximal in \mathcal{V} if $\mathcal{O}(N) = \mathcal{O}(M)$ for each $N \in \mathcal{V}$ such that $\mathcal{O}(M) \subseteq \overline{\mathcal{O}(N)}$.

Put $a_{\mathbf{\Delta}}(\mathbf{d}) := \dim \text{GL}(\mathbf{d}) - q_{\mathbf{\Delta}}(\mathbf{d})$ for a bound quiver $\mathbf{\Delta}$ and a dimension vector \mathbf{d} . The following facts were proved in [7].

Proposition 3.2. *Let \mathbf{d} be the dimension vector of a periodic representation over a tame concealed-canonical bound quiver $\mathbf{\Delta}$. Then the following hold.*

- (1) *The variety $\text{rep}_{\mathbf{\Delta}}(\mathbf{d})$ is a normal complete intersection of dimension $a_{\mathbf{\Delta}}(\mathbf{d})$.*
- (2) *If there exists $M \in \text{rep}_{\mathbf{\Delta}}(\mathbf{d})$ such that $\text{Ext}_{\mathbf{\Delta}}^1(M, M) = 0$, then $\overline{\mathcal{O}(M)} = \text{rep}_{\mathbf{\Delta}}(\mathbf{d})$.*
- (3) *If $\text{Ext}_{\mathbf{\Delta}}^1(M, M) \neq 0$ for each $M \in \text{rep}_{\mathbf{\Delta}}(\mathbf{d})$, then there exists a convex subquiver $\mathbf{\Delta}'$ of $\mathbf{\Delta}$ and a sincere separating exact subcategory \mathcal{R}' in $\text{rep } \mathbf{\Delta}'$ such that $M \in \text{add } \mathcal{R}'$ for each maximal orbit $\mathcal{O}(M)$ in $\text{rep}_{\mathbf{\Delta}}(\mathbf{d})$.*
- (4) *If $M \in \text{rep}_{\mathbf{\Delta}}(\mathbf{d})$, then there is a canonical epimorphism*

$$\pi_M : T_M \text{rep}_{\mathbf{\Delta}}(\mathbf{d}) \rightarrow \text{Ext}_{\mathbf{\Delta}}^1(M, M)$$

with kernel $T_M \mathcal{O}(M)$. \square

Let \mathbf{d} be the dimension vector of a periodic module over a tame concealed-canonical bound quiver $\mathbf{\Delta}$. The above theorem implies that in order to prove that $\overline{\mathcal{O}(M)}$ is a normal complete intersection for each maximal orbit $\mathcal{O}(M)$ in $\text{rep}_{\mathbf{\Delta}}(\mathbf{d})$, we may assume that \mathbf{d} is the

dimension vector of a direct sum of modules from a sincere separating exact subcategory of $\text{ind } \Delta$. Thus we fix a tame bound quiver Δ and a sincere separating exact subcategory \mathcal{R} of $\text{ind } \Delta$. We will use freely notation introduced in Section 2. It follows from [7, Section 3] that if $\mathbf{d} \in \mathbf{R}$, then $M \in \text{add } \mathcal{R}$ for each maximal orbit $\mathcal{O}(M)$ in $\text{rep}_\Delta(\mathbf{d})$.

For a full subcategory \mathcal{X} of $\text{ind } \Delta$ and a dimension vector \mathbf{d} we denote by $\mathcal{X}(\mathbf{d})$ the intersection of $\text{rep}_\Delta(\mathbf{d})$ with $\text{add } \mathcal{X}$. If $\mathbf{d}', \mathbf{d}'' \in \mathbb{N}^{\Delta_0}$, $C' \subseteq \text{rep}_\Delta(\mathbf{d}')$ and $C'' \subseteq \text{rep}_\Delta(\mathbf{d}'')$, then we denote by $C' \oplus C''$ the subset of $\text{rep}_\Delta(\mathbf{d}' + \mathbf{d}'')$ consisting of all M such that $M \simeq M' \oplus M''$ for some $M' \in C'$ and $M'' \in C''$. The following fact follows from [4, Section 3].

Proposition 3.3. *If $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$, then $(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$ is an irreducible constructible subset of $\text{rep}_\Delta(\mathbf{d}' + \mathbf{d}'')$ of dimension $a_\Delta(\mathbf{d}) + \langle \mathbf{d}'', \mathbf{d}' \rangle_\Delta$. \square*

Using Corollary 2.2 we immediately get the following.

Corollary 3.4. *Let $\mathbf{d} \in \mathbf{R}$, $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$. If $p^{\mathbf{d}} > 0$, $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$ and $\mathbf{d}'' \neq 0$, then*

$$\dim((\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')) \leq a_\Delta(\mathbf{d}) - p^{\mathbf{d}} - 1.$$

Moreover, the equality holds if and only if one of the following conditions is satisfied:

- (1) $q_\Delta(\mathbf{d}'') = 1$ and $\langle \mathbf{d}'', \mathbf{d} \rangle_\Delta = -p^{\mathbf{d}}$ (in particular, $\langle \mathbf{d}'', \mathbf{h} \rangle_\Delta = -1$),
or
- (2) $q_\Delta(\mathbf{d}'') = 0$ and $\langle \mathbf{d}'', \mathbf{d} \rangle_\Delta = -2$ (in particular, Δ is of type $(2, 2, 2, 2)$ and $\mathbf{d} = \mathbf{h}$). \square

Observe that

$$\text{rep}_\Delta(\mathbf{d}) = \mathcal{R}(\mathbf{d}) \cup \bigcup_{\substack{\mathbf{d}' \in \mathbf{P} + \mathbf{R}, \mathbf{d}'' \in \mathbf{Q} \\ \mathbf{d}' + \mathbf{d}'' = \mathbf{d}, \mathbf{d}'' \neq 0}} (\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$$

for each $\mathbf{d} \in \mathbf{R}$. Indeed, if $M \in (\mathcal{P} \cup \mathcal{R})(\mathbf{d})$ and we write $M = M' \oplus M''$ for $M' \in \text{add } \mathcal{P}$ and $M'' \in \text{add } \mathcal{R}$, then $\langle \dim M', \mathbf{h} \rangle_\Delta = \langle \mathbf{d}, \mathbf{h} \rangle_\Delta = 0$, hence $M' = 0$ by Proposition 2.1(3). The above formula together with Corollary 3.4 implies that $\dim(\text{rep}_\Delta(\mathbf{d}) \setminus \mathcal{R}(\mathbf{d})) \leq a_\Delta(\mathbf{d}) - p^{\mathbf{d}} - 1$.

4. STABILITY AND SEMI-INVARIANTS

Let Δ be a quiver and $\theta \in \mathbb{Z}^{\Delta_0}$. We treat θ as a \mathbb{Z} -linear function $\mathbb{Z}^{\Delta_0} \rightarrow \mathbb{Z}$ in a usual way. A representation M of Δ is called θ -semi-stable if $\theta(\dim M) = 0$ and $\theta(\dim N) \geq 0$ for each subrepresentation N of M . The full subcategory of θ -semi-stable representations of Δ is an exact subcategory of $\text{rep } \Delta$. Two θ -semi-stable representations are called S-equivalent if they have the same composition factors within this category. If \mathbf{d} is a dimension vector, then by a semi-invariant of weight θ we mean every function $c \in k[\text{rep}_\Delta(\mathbf{d})]$ such that $c(g \cdot M) =$

$\chi^\theta(g)c(M)$ for any $g \in \mathrm{GL}(\mathbf{d})$ and $M \in \mathrm{rep}_\Delta(\mathbf{d})$, where $\chi^\theta(g) := \prod_{x \in \Delta_0} (\det g(x))^{\theta(x)}$ for $g \in \mathrm{GL}(\mathbf{d})$.

Now let Δ be a bound quiver and \mathbf{d} a dimension vector. If $\theta \in \mathbb{Z}^{\Delta_0}$, then a function $c \in k[\mathrm{rep}_\Delta(\mathbf{d})]$ is called a semi-invariant of weight θ if c is a restriction of a semi-invariant of weight θ from $k[\mathrm{rep}_\Delta(\mathbf{d})]$. This definition differs from the definition used in other papers on the subject (see for example [5, 18–20]), however it is consistent with King's approach [24]. We denote the space of the semi-invariants of weight θ by $\mathrm{SI}[\Delta, \mathbf{d}]_\theta$. If $\theta \in \mathbb{Z}^{\Delta_0}$, then we put $\Lambda_\theta(\mathbf{d}) := \bigoplus_{n \in \mathbb{N}} \mathrm{SI}[\Delta, \mathbf{d}]_{n\theta}$. Note that $\Lambda_\theta(\mathbf{d})$ is a graded ring. For $M \in \mathrm{rep}_\Delta(\mathbf{d})$ we denote by $\mathcal{I}_\theta(M)$ the ideal in $\Lambda_\theta(\mathbf{d})$ generated by the homogeneous elements c such that $c(M) = 0$.

The following results were proved in [24].

Proposition 4.1. *Let Δ be a bound quiver, \mathbf{d} a dimension vector, and $\theta \in \mathbb{Z}^{\Delta_0}$.*

- (1) *If $M \in \mathrm{rep}_\Delta(\mathbf{d})$, then M is θ -semi-stable if and only if there exists a semi-invariant c of weight $n\theta$ for some $n \in \mathbb{N}_+$ such that $c(M) \neq 0$.*
- (2) *If $M, N \in \mathrm{rep}_\Delta(\mathbf{d})$ are θ -semi-stable, then M and N are S-equivalent if and only if $\mathcal{I}_\theta(M) = \mathcal{I}_\theta(N)$. \square*

Now we recall a construction from [19]. Let Δ be a bound quiver. Fix a representation V of Δ . We define $\theta^V : \mathbb{Z}^{\Delta_0} \rightarrow \mathbb{Z}$ by the condition

$$\theta^V(\mathbf{dim} M) = -\dim_k \mathrm{Hom}_\Delta(V, M) + \dim_k \mathrm{Hom}_\Delta(M, \tau V)$$

for each representation M of Δ . The Auslander–Reiten formula implies that $\theta^V = -\langle \mathbf{dim} V, - \rangle$ if $\mathrm{pdim}_\Delta V \leq 1$. Dually, if $\mathrm{idim}_\Delta V \leq 1$, then $\theta^V = \langle -, \mathbf{dim} \tau V \rangle$.

Now let \mathbf{d} be a dimension vector. If $\theta^V(\mathbf{d}) = 0$, then we define a function $c^V \in k[\mathrm{rep}_\Delta(\mathbf{d})]$ in the following way. Let $P_1 \xrightarrow{f} P_0 \rightarrow V \rightarrow 0$ be a minimal projective presentation of V . There exist vertices $x_1, \dots, x_n, y_1, \dots, y_m$ of Δ such that $P_1 = \bigoplus_{i \in [1, n]} P_{x_i}$ and $P_0 = \bigoplus_{j \in [1, m]} P_{y_j}$. Moreover, there exist $\omega_{i,j} \in k\Delta(y_j, x_i)$, $i \in [1, n]$, $j \in [1, m]$, such that $f = [P_{\omega_{i,j}}]_{\substack{j \in [1, m] \\ i \in [1, n]}}$. Consequently, if $M \in \mathrm{rep}_\Delta(\mathbf{d})$, then

$$\mathrm{Hom}_\Delta(f, M) = [M(\omega_{i,j})]_{\substack{i \in [1, n] \\ j \in [1, m]}} : \bigoplus_{j \in [1, m]} M(y_j) \rightarrow \bigoplus_{i \in [1, n]} M(x_i).$$

In addition, one shows $\dim_k \mathrm{Ker} \mathrm{Hom}_\Delta(f, M) = \dim_k \mathrm{Hom}_\Delta(V, M)$ and $\dim_k \mathrm{Coker} \mathrm{Hom}_\Delta(f, M) = \dim_k \mathrm{Hom}_\Delta(M, \tau V)$. Consequently,

$$\begin{aligned} & \sum_{j \in [1, m]} \dim_k M(y_j) - \sum_{i \in [1, n]} \dim_k M(x_i) \\ &= \dim_k \mathrm{Hom}_\Delta(M, \tau V) - \dim_k \mathrm{Hom}_\Delta(V, M) = \theta^V(\mathbf{d}) = 0. \end{aligned}$$

Thus, it makes sense to define $c^V \in k[\text{rep}_\Delta(\mathbf{d})]$ by

$$c^V(M) := \det \text{Hom}_\Delta(f, M)$$

for $M \in \text{rep}_\Delta(\mathbf{d})$. Note that $c^V(M) = 0$ if and only if $\text{Hom}_\Delta(V, M) \neq 0$. It is known that $c^V \in \text{SI}[\Delta, \mathbf{d}]_{\theta^V}$. This function depends on the choice of f , but functions obtained for different f 's differ only by non-zero scalars. In fact, we could start with an arbitrary projective presentation $P_1 \xrightarrow{f} P_0 \rightarrow V \rightarrow 0$ of V such that $\dim_k \text{Hom}_\Delta(P_1, M) = \dim_k \text{Hom}_\Delta(P_0, M)$. As an easy consequence we obtain the following (see [18, Proposition 2] and [19, Lemma 3.3]).

Lemma 4.2. *Let Δ be a bound quiver and \mathbf{d} a dimension vector.*

- (1) *If $V = V_1 \oplus V_2$, $\theta^V(\mathbf{d}) = 0$ and $c^V \neq 0$, then $\theta^{V_1}(\mathbf{d}) = 0 = \theta^{V_2}(\mathbf{d})$.*
- (2) *If $0 \rightarrow V_1 \rightarrow V \rightarrow V_2 \rightarrow 0$ and $\theta^V(\mathbf{d}) = \theta^{V_1}(\mathbf{d}) = \theta^{V_2}(\mathbf{d}) = 0$, then $c^V = c^{V_1}c^{V_2}$.* \square

The following result follows from the proof of [19, Theorem 3.2] (note that the assumption about the characteristic of k made in [19, Theorem 3.2] is only necessary for surjectivity of the restriction morphism, which we have for free with our definition of semi-invariants).

Proposition 4.3. *Let Δ be a bound quiver and \mathbf{d} a dimension vector. If $\theta \in \mathbb{Z}^{\Delta_0}$, then the space $\text{SI}[\Delta, \mathbf{d}]_\theta$ is spanned by the functions c^V for $V \in \text{rep}_\Delta$ such that $\theta^V = \theta$.* \square

Now we apply our considerations in the case of tame concealed-canonical quivers. For the rest of the section we fix a tame bound quiver Δ and a sincere separating exact subcategory \mathcal{R} of $\text{ind } \Delta$. We will use notation introduced in Section 2. We fix $\mathbf{d} \in \mathbf{R}$ such that $p^{\mathbf{d}} > 0$ and put $\theta := -\langle \mathbf{h}, - \rangle_\Delta$.

First observe that $M \in \text{rep } \Delta$ is θ -semi-stable if and only if $M \in \text{add } \mathcal{R}$. Consequently, if M and N are θ -semi-stable, then M and N are S-equivalent if and only if $q_{\lambda,i}^M = q_{\lambda,i}^N$ for any $\lambda \in \mathbb{X}$ and $i \in [0, r_\lambda - 1]$. In particular, there are only finitely many isomorphism classes in each S-equivalence class.

Now fix $V \in \text{rep } \Delta$ such that $\theta^V = n\theta$ for some $n \in \mathbb{N}$ and $c^V \neq 0$. We show that $V \in \text{add } \mathcal{R}$ and $\dim V = n\mathbf{h}$. Indeed, write $V = P \oplus R \oplus Q$ for $P \in \text{add } \mathcal{P}$, $R \in \text{add } \mathcal{R}$ and $Q \in \text{add } \mathcal{Q}$. If $P \neq 0$, then $\theta^P(\mathbf{d}) \leq -\langle \dim P, \mathbf{h} \rangle_\Delta < 0$ by Proposition 2.1(3), hence $c^V = 0$ by Lemma 4.2(1). Consequently, $P = 0$ and, dually, $Q = 0$, thus $V = R \in \text{add } \mathcal{R}$. In particular, $\text{pdim}_\Delta V = 1$, hence $-\langle n\mathbf{h}, - \rangle = \theta^V = -\langle \dim V, - \rangle_\Delta$, and this implies that $\dim V = n\mathbf{h}$.

For $\lambda \in \mathbb{X}$ we denote by $\mathcal{A}_\lambda(\mathbf{d})$ the set of all $i \in [0, r_\lambda - 1]$ such that $p_{\lambda,i}^{\mathbf{d}} = 0$. Next, for $i \in \mathcal{A}_\lambda(\mathbf{d})$ we denote by $n_{\lambda,i}$ the minimal $n \in \mathbb{N}_+$ such that $p_{\lambda,(i-n) \bmod r_\lambda}^{\mathbf{d}} = 0$, and put $V_{\lambda,i} := R_{\lambda,i}^{(n_{\lambda,i})}$. Observe that $\theta^{V_{\lambda,i}}(\mathbf{d}) = -\langle \dim V_{\lambda,i}, \mathbf{d} \rangle_\Delta = 0$ for any $\lambda \in \mathbb{X}$ and $i \in \mathcal{A}_\lambda(\mathbf{d})$. We

put $c_{\lambda,i} := c^{V_{\lambda,i}}$ for $\lambda \in \mathbb{X}$ and $i \in \mathcal{A}_\lambda(\mathbf{d})$. More generally, if $\lambda \in \mathbb{X}$ and $J \subseteq \mathcal{A}_\lambda(\mathbf{d})$, then we put $V_{\lambda,J} := \bigoplus_{i \in J} V_{\lambda,i}$ and $c_{\lambda,J} := c^{V_{\lambda,J}} = \prod_{i \in J} c_{\lambda,i}$. In particular, we put $V_\lambda := V_{\lambda, \mathcal{A}_\lambda(\mathbf{d})}$ and $c_\lambda := c_{\lambda, \mathcal{A}_\lambda(\mathbf{d})}$ for $\lambda \in \mathbb{X}$. Then $c_\lambda \in \text{SI}[\Delta, \mathbf{d}]_\theta$ for each $\lambda \in \mathbb{X}$. Observe that Lemma 4.2(2) implies that $c_\lambda = c^{R_{\lambda,i}^{(r_\lambda)}}$ for any $\lambda \in \mathbb{X}$ and $i \in \mathcal{A}_\lambda(\mathbf{d})$. More general, $c_\lambda^p = c^{R_{\lambda,i}^{(pr_\lambda)}}$ for any $p \in \mathbb{N}_+$, $\lambda \in \mathbb{X}$ and $i \in \mathcal{A}_\lambda(\mathbf{d})$.

We have the following information about $\Lambda_\theta(\mathbf{d})$.

Proposition 4.4. *Let $\mathbf{d} \in \mathbf{R}$ be such that $p^\mathbf{d} > 0$ and $\theta := -\langle \mathbf{h}, - \rangle$. Then $\Lambda_\theta(\mathbf{d})$ is generated by the functions c_λ , $\lambda \in \mathbb{X}$.*

Proof. First we show that if $\lambda \in \mathbb{X}$, $i \in \mathbb{Z}$, $n \in \mathbb{N}_+$, $\theta^{R_{\lambda,i}^{(n)}}(\mathbf{d}) = 0$ and $c^{R_{\lambda,i}^{(n)}} \neq 0$, then $p_{\lambda,i \bmod r_\lambda}^\mathbf{d} = p_{\lambda,(i-n) \bmod r_\lambda}^\mathbf{d}$ and $p_{\lambda,j \bmod r_\lambda}^\mathbf{d} \geq p_{\lambda,i \bmod r_\lambda}^\mathbf{d}$ for each $j \in [i-n+1, i-1]$. Indeed, the former condition follows from the equality $\langle \mathbf{e}_{\lambda,i}^{(n)}, \mathbf{d} \rangle = -\theta^{R_{\lambda,i}^{(n)}}(\mathbf{d}) = 0$. Moreover, if there exists $j \in [i-n+1, i-1]$ such that $p_{\lambda,j \bmod r_\lambda}^\mathbf{d} < p_{\lambda,i \bmod r_\lambda}^\mathbf{d}$, then $\text{Hom}_\Delta(R_{\lambda,i}^{(n)}, R) \neq 0$ for each $R \in \mathcal{R}(\mathbf{d})$, hence $c^{R_{\lambda,i}^{(n)}} = 0$.

We have the following important consequence of the above observation. Assume that $\lambda \in \mathbb{X}$, $i \in [0, r_\lambda - 1]$, $p \in \mathbb{N}_+$ and $c^{R_{\lambda,i}^{(pr_\lambda)}} \neq 0$. Then $p_{\lambda,j}^\mathbf{d} \geq p_{\lambda,i}^\mathbf{d}$ for each $j \in [0, r_\lambda - 1]$, hence $i \in \mathcal{A}_\lambda(\mathbf{d})$. In particular, $c^{R_{\lambda,i}^{(pr_\lambda)}} = c_\lambda^p$.

Now assume that $R \in \text{rep } \Delta$, $\theta^R = n\theta$ for some $n \in \mathbb{N}$, and $c^R \neq 0$. We know that $R \in \text{add } \mathcal{R}$ and $\mathbf{dim } R = n\mathbf{h}$. If $R = \bigoplus_{\lambda \in \mathbb{X}} R_\lambda$ for $R_\lambda \in \mathcal{R}_\lambda$, $\lambda \in \mathbb{X}$, then $\mathbf{dim } R_\lambda = p_\lambda^R \mathbf{h}$ for each $\lambda \in \mathbb{X}$. We show that $c^{R_\lambda} = c_\lambda^{p_\lambda^R}$ for each $\lambda \in \mathbb{X}$, hence the claim will follow from Lemma 4.2(1).

Fix $\lambda \in \mathbb{X}$ and write $R_\lambda = \bigoplus_{j \in [1, m]} R_{\lambda, i_j}^{(n_j)}$ for $m \in \mathbb{N}_+$, $i_1, \dots, i_m \in \mathbb{Z}$ and $n_1, \dots, n_m \in \mathbb{N}_+$. If $n_j \equiv 0 \pmod{r_\lambda}$ for each $j \in [1, m]$, then the claim follows. Thus assume $n_1 \not\equiv 0 \pmod{r_\lambda}$. Since $\mathbf{dim } R_\lambda = p_\lambda^R \mathbf{h}$, we may assume that $i_2 = i_1 - n_1$. Then we have an exact sequence $0 \rightarrow R_{\lambda, i_2}^{(n_2)} \rightarrow R_{\lambda, i_1}^{(n_1+n_2)} \rightarrow R_{\lambda, i_1}^{(n_1)} \rightarrow 0$, hence Lemma 4.2(2) implies that $c^R = c^{R'}$, where $R' := R_{\lambda, i_1}^{(n_1+n_2)} \oplus \bigoplus_{j \in [3, m]} R_{\lambda, i_j}^{(n_j)}$. Now the claim follows by induction. \square

As a consequence we get the following.

Corollary 4.5. *Let $\mathbf{d} \in \mathbf{R}$ be such that $p^\mathbf{d} > 0$ and $\theta := -\langle \mathbf{h}, - \rangle$. If $M, N \in \mathcal{R}(\mathbf{d})$, then M and N are S -equivalent if and only if there exists $\mu \in k$ such that $c_\lambda(M) = \mu c_\lambda(N)$ for each $\lambda \in \mathbb{X}$.*

Proof. Follows immediately from Propositions 4.1(2) and 4.4. \square

We list some consequences of the description of the maximal orbits in $\text{rep}_\Delta(\mathbf{d})$ given in [7, Proposition 5] (see also [30, Theorem 3.5]). Recall that $M \in \mathcal{R}(\mathbf{d})$ for each maximal orbit $\mathcal{O}(M)$ in $\text{rep}_\Delta(\mathbf{d})$. Next,

if $\mathcal{O}(M)$ is maximal in $\text{rep}_\Delta(\mathbf{d})$, then $\dim \mathcal{O}(M) = a_\Delta(\mathbf{d}) - p^{\mathbf{d}}$. In particular, the maximal orbits in $\text{rep}_\Delta(\mathbf{d})$ coincide with the orbits of maximal dimension. Moreover, if $\lambda \in \mathbb{X}$, then there exists at most one $i \in \mathcal{A}_\lambda(\mathbf{d})$ such that $c_{\lambda,i}(M) = 0$. We put

$$\hat{\mathbb{X}}(M) := \{(\lambda, i) : \lambda \in \mathbb{X}, i \in \mathcal{A}_\lambda(\mathbf{d}) \text{ and } c_{\lambda,i}(M) = 0\},$$

and denote by $\mathbb{X}(M)$ the image of $\hat{\mathbb{X}}(M)$ under the projection on the first coordinate. If $\lambda \in \mathbb{X}$, then $\lambda \in \mathbb{X}(M)$ if and only if $p_\lambda^M \neq 0$. In particular, $|\mathbb{X}(M)| \leq p^{\mathbf{d}}$. Finally, if $M, N \in \text{rep}_\Delta(\mathbf{d})$ are S-equivalent, the orbits $\mathcal{O}(M)$ and $\mathcal{O}(N)$ are maximal, and $\hat{\mathbb{X}}(M) \subseteq \hat{\mathbb{X}}(N)$, then $\mathcal{O}(M) = \mathcal{O}(N)$.

For a representation V of Δ such that $\theta^V(\mathbf{d}) = 0$ we denote by $\mathcal{H}^V(\mathbf{d})$ the zero set of c^V , i.e. $\mathcal{H}^V(\mathbf{d}) := \{M \in \text{rep}_\Delta(\mathbf{d}) : \text{Hom}_\Delta(V, M) \neq 0\}$. Moreover, we say that an exact sequence $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ is V -exact if the induced sequence

$$0 \rightarrow \text{Hom}_\Delta(V, M) \rightarrow \text{Hom}_\Delta(V, N) \rightarrow \text{Hom}_\Delta(V, L) \rightarrow 0$$

is exact. We need the following version of [29, Corollary 7.4].

Proposition 4.6. *Let V be a representation of Δ such that $\theta^V(\mathbf{d}) = 0$.*

(1) *If $M \in \mathcal{H}^V(\mathbf{d})$ and $\dim_k \text{Hom}_\Delta(V, M) = 1$, then*

$$\text{Ker } \partial c^V(M) = \{Z \in T_M \text{rep}_\Delta(\mathbf{d}) : \pi_M(Z) \text{ is } V\text{-exact}\}.$$

(2) *If $M \in \mathcal{H}^V(\mathbf{d})$ and $\dim_k \text{Hom}_\Delta(V, M) \geq 2$, then $\text{Ker } \partial c^V(M) = T_M \text{rep}_\Delta(\mathbf{d})$.*

5. AUXILIARY LEMMAS

Throughout this section we fix a tame bound quiver Δ and a sincere separating exact subcategory \mathcal{R} of $\text{ind } \Delta$. We use freely notation introduced in Section 2. We also fix $\mathbf{d} \in \mathbf{R}$ such that $p := p^{\mathbf{d}} > 0$.

Lemma 5.1. *If $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ are pairwise different, then*

$$\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d}) = \bigcap_{\lambda \in \mathbb{X}} \mathcal{H}^{V_\lambda}(\mathbf{d}) = \bigcup_{\substack{\mathbf{d}' \in \mathbf{P} + \mathbf{R}, \mathbf{d}'' \in \mathbf{Q} \\ \mathbf{d}' + \mathbf{d}'' = \mathbf{d}, \mathbf{d}'' \neq 0}} (\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'').$$

Proof. Obviously, $\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d}) \supseteq \bigcap_{\lambda \in \mathbb{X}} \mathcal{H}^{V_\lambda}(\mathbf{d})$.

Now fix $\lambda \in \mathbb{X}$, $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$ such that $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$ and $\mathbf{d}'' \neq 0$. If $P \in \mathcal{P}(\mathbf{d}')$ and $Q \in \mathcal{Q}(\mathbf{d}'')$, then Proposition 2.1(3) implies that

$$\dim_k \text{Hom}_\Delta(V_\lambda, P \oplus Q) \geq \dim_k \text{Hom}_\Delta(V_\lambda, Q) = \langle \mathbf{h}, \mathbf{d}'' \rangle_\Delta > 0,$$

hence $(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'') \subseteq \mathcal{H}^{V_\lambda}(\mathbf{d})$.

Finally, assume that $R \in \mathcal{R}(\mathbf{d}) \cap \bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$. Then $p_{\lambda_l}^R > 0$ for each $l \in [0, p]$. Consequently, $p^R \geq \sum_{l \in [0, p]} p_{\lambda_l}^R > p$, a contradiction. \square

Corollary 5.2. *Let $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ be pairwise different. If \mathcal{C} is an irreducible component of $\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$, then $\dim \mathcal{C} = a_{\Delta}(\mathbf{d}) - p - 1$ and there exist $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$ such that $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$, $\mathbf{d}'' \neq 0$ and $\mathcal{C} = \overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$.*

Proof. It follows from Lemma 5.1 that \mathcal{C} is an irreducible component of $\overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ for some $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$ such that $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$ and $\mathbf{d}'' \neq 0$. Since $\overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ is irreducible by Proposition 3.3, $\mathcal{C} = \overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$.

We know from Proposition 3.2(1) that $\dim \text{rep}_{\Delta}(\mathbf{d}) = a_{\Delta}(\mathbf{d})$, hence Krull's Principal Ideal Theorem [25, Section V.3] implies that $\dim \mathcal{C} \geq a_{\Delta}(\mathbf{d}) - p - 1$. On the other hand, $\dim \mathcal{C} = \dim \overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ $\leq a_{\Delta}(\mathbf{d}) - p - 1$ by Corollary 3.4, and the claim follows. \square

Lemma 5.3. *Let $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ be pairwise different and $J_l \subseteq \mathcal{A}_{\lambda_l}(\mathbf{d})$, $l \in [0, p]$. If \mathcal{C} is an irreducible component of $\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l, J_l}}(\mathbf{d})$, then \mathcal{C} is an irreducible component of $\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$.*

Proof. Similarly as in the proof of Corollary 5.2 we show that $\dim \mathcal{C} \geq a_{\Delta}(\mathbf{d}) - p - 1$. On the other hand, $\mathcal{C} \subseteq \bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$, hence there exists an irreducible component \mathcal{C}' of $\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$ such that $\mathcal{C} \subseteq \mathcal{C}'$. Corollary 5.2 says that $\dim \mathcal{C}' = a_{\Delta}(\mathbf{d}) - p - 1$, hence $\mathcal{C} = \mathcal{C}'$. \square

Corollary 5.4. *Let $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ be pairwise different and $J_l \subseteq \mathcal{A}_{\lambda_l}(\mathbf{d})$, $l \in [0, p]$. If \mathcal{C} is an irreducible component of $\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l, J_l}}(\mathbf{d})$, then $\dim \mathcal{C} = a_{\Delta}(\mathbf{d}) - p - 1$ and there exist $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$ such that $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$, $\mathbf{d}'' \neq 0$ and $\mathcal{C} = \overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$.*

Proof. Immediate from Lemma 5.3 and Corollary 5.2. \square

Proposition 5.5. *Let $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ be pairwise different and $J_l \subseteq \mathcal{A}_{\lambda_l}(\mathbf{d})$, $l \in [0, p]$. If $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$, $\mathbf{d}'' \in \mathbf{Q}$ and $\overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ is an irreducible component of $\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l, J_l}}(\mathbf{d})$, then $\langle \dim V_{\lambda_l, J_l}, \mathbf{d}'' \rangle_{\Delta} > 0$ for each $l \in [0, p]$. Moreover, if $\langle \dim V_{\lambda_l, J_l}, \mathbf{d}'' \rangle_{\Delta} = 1$ for each $l \in [0, p]$, then there exists $M \in \overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ such that $\partial c_{\lambda_0, J_0}(M), \dots, \partial c_{\lambda_p, J_p}(M)$ are linearly independent.*

Proof. We know from Lemma 5.3 that $\overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ is an irreducible component of $\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$. Fix $M \in \overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ such that $\mathcal{O}(M)$ is maximal in $\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$. Write $M = P \oplus Q$ for $P \in \overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d})}$ and $Q \in \mathcal{Q}(\mathbf{d}'')$.

First we prove that $\text{Hom}_{\Delta}(V_{\lambda_l, J_l}, P) = 0$ for each $l \in [0, p]$. This will imply in particular that

$$\langle \dim V_{\lambda_l, J_l}, \mathbf{d}'' \rangle_{\Delta} = \dim_k \text{Hom}_{\Delta}(V_{\lambda_l, J_l}, Q) = \dim_k \text{Hom}_{\Delta}(V_{\lambda_l, J_l}, M) > 0$$

for each $l \in [0, p]$. Write $P = P' \oplus R$ for $P' \in \text{add } \mathcal{P}$ and $R \in \text{add } \mathcal{R}$, and assume $\text{Hom}_{\Delta}(V_{\lambda_l, i}, R) \neq 0$ for some $l \in [0, p]$ and $i \in J_l$. Then

$q_{\lambda_l, i}^R > 0$. If $p^R > 0$, then $\langle \mathbf{dim} Q, \mathbf{dim} R \rangle_{\Delta} \leq \langle \mathbf{d}'', \mathbf{h} \rangle_{\Delta} < 0$ by Proposition 2.1(3) (recall that $\mathbf{d}'' \neq 0$ by Corollary 5.4). Otherwise, we fix $n \in \mathbb{N}$ such that $q_{\lambda, (i+n) \bmod r_{\lambda}}^R = 0$ and $q_{\lambda, (i+j) \bmod r_{\lambda}}^R > 0$ for each $j \in [1, n-1]$. Then

$$\begin{aligned} \langle \mathbf{d}'', \mathbf{e}_{\lambda_l, i+n-1}^n \rangle_{\Delta} &= \langle \mathbf{d} - \mathbf{dim} P' - \mathbf{dim} R, \mathbf{e}_{\lambda_l, i+n-1}^n \rangle_{\Delta} \\ &\leq -p_{\lambda_l, (i+n) \bmod r_{\lambda}}^{\mathbf{d}} - \langle \mathbf{dim} P', \mathbf{e}_{\lambda_l, i+n-1}^n \rangle_{\Delta} - q_{\lambda_l, i}^R < 0. \end{aligned}$$

This again implies that $\langle \mathbf{dim} Q, \mathbf{dim} R \rangle_{\Delta} < 0$, hence $\text{Ext}_{\Delta}^1(Q, R) \neq 0$. If $0 \rightarrow R \rightarrow Q' \rightarrow Q \rightarrow 0$ is a non-split exact sequence, then $P' \oplus Q' \in \bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$, since $\dim_k \text{Hom}_{\Delta}(V_{\lambda}, Q') \geq \langle \mathbf{h}, \mathbf{dim} Q' \rangle_{\Delta} = \langle \mathbf{h}, \mathbf{d}'' \rangle_{\Delta} > 0$ for each $\lambda \in \mathbb{X}$. Moreover, $M \in \overline{\mathcal{O}(P' \oplus Q')}$ and $M \not\cong P' \oplus Q'$, which contradicts the maximality of $\mathcal{O}(M)$.

Now we assume that $\langle \mathbf{dim} V_{\lambda_l, J_l}, \mathbf{d}'' \rangle_{\Delta} = 1$ for each $l \in [0, p]$ and prove that under this assumption $\partial c_{\lambda_0, J_0}(M), \dots, \partial c_{\lambda_p, J_p}(M)$ are linearly independent. Our assumption implies that

$$\dim_k \text{Hom}_{\Delta}(V_{\lambda_l, J_l}, M) = \dim_k \text{Hom}_{\Delta}(V_{\lambda_l, J_l}, Q) = 1$$

for each $l \in [0, p]$. Let $K := \bigcap_{l \in [0, p]} \partial c^{V_{\lambda_l, J_l}}(M) \subseteq T_M \text{rep}_{\Delta}(\mathbf{d})$. We have the canonical inclusion $\text{Ext}_{\Delta}^1(Q, P) \hookrightarrow \text{Ext}_{\Delta}^1(M, M)$, which sends an exact sequence $\xi : 0 \rightarrow P \rightarrow N \rightarrow Q \rightarrow 0$ to the sequence

$$\xi' : 0 \rightarrow M \rightarrow N \oplus M \rightarrow M \rightarrow 0.$$

Using Proposition 4.6(1) we obtain that $\xi' \in \pi_M(K)$ if and only if $\dim_k \text{Hom}_{\Delta}(V_{\lambda_l, J_l}, N) = 1$ for each $l \in [0, p]$. In particular, this implies that $N \in \bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l, J_l}}(\mathbf{d})$. By the maximality of $\mathcal{O}(M)$, $N \simeq M$, i.e. ξ splits, thus Proposition 3.2(4) implies

$$\text{codim}_{T_M \text{rep}_{\Delta}(\mathbf{d})} K \geq \dim_k \text{Ext}_{\Delta}^1(Q, P) \geq -\langle \mathbf{d}'', \mathbf{d}' \rangle_{\Delta}.$$

It follows from Corollary 2.2 that $-\langle \mathbf{d}'', \mathbf{d}' \rangle_{\Delta} \geq p+1$, and this finishes the proof. \square

Let $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ be pairwise different and $J_l \subseteq \mathcal{A}_{\lambda_l}(\mathbf{d})$, $l \in [0, p]$. Assume that $\overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ is an irreducible component of $\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l, J_l}}(\mathbf{d})$ for $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$. We know from Corollary 5.4 that $\dim(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'') = a_{\Delta}(\mathbf{d}) - p - 1$. Consequently, either $q_{\Delta}(\mathbf{d}'') = 0$ or $q_{\Delta}(\mathbf{d}'') = 1$ by Corollary 3.4. We prove that in the latter case there is always $M \in (\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$ such that $\partial c_{\lambda_0, J_0}(M), \dots, \partial c_{\lambda_p, J_p}(M)$ are linearly independent.

Corollary 5.6. *Let $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ be pairwise different and $J_l \subseteq \mathcal{A}_{\lambda_l}(\mathbf{d})$, $l \in [0, p]$. If $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$, $\mathbf{d}'' \in \mathbf{Q}$, $\overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ is an irreducible component of $\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l, J_l}}(\mathbf{d})$, and $q_{\Delta}(\mathbf{d}'') = 1$, then there exists $M \in (\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$ such that $\partial c_{\lambda_0, J_0}(M), \dots, \partial c_{\lambda_p, J_p}(M)$ are linearly independent.*

Proof. From the previous proposition we know that $\langle \mathbf{dim} V_{\lambda_l, J_l}, \mathbf{d}'' \rangle_{\Delta} > 0$ for each $l \in [0, p]$. On the other hand, Corollary 3.4 implies that $\langle \mathbf{dim} V_{\lambda_l, J_l}, \mathbf{d}'' \rangle_{\Delta} \leq \langle \mathbf{h}, \mathbf{d}'' \rangle_{\Delta} = 1$ for each $l \in [0, p]$. Consequently, $\langle \mathbf{dim} V_{\lambda_l, J_l}, \mathbf{d}'' \rangle_{\Delta} = 1$ for each $l \in [0, p]$, and the claim follows from the previous proposition. \square

6. NONSINGULAR DIMENSION VECTORS

Throughout this section we fix a sincere separating exact subcategory \mathcal{R} of $\text{ind } \Delta$ for a tame bound quiver Δ and use freely notation introduced in Section 2. We also fix $\mathbf{d} \in \mathbf{R}$ such that $p := p^{\mathbf{d}} > 0$. Finally, we assume that \mathbf{d} is not singular. This assumption implies, according to Proposition 2.3(2) and Corollary 3.4, that $q_{\Delta}(\mathbf{d}'') = 1$ for any $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$ such that $\mathbf{d}' + \mathbf{d}'' = \mathbf{d}$ and $\dim(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'') = a_{\Delta}(\mathbf{d}) - p - 1$. Consequently, we have the following.

Lemma 6.1. *Let $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ be pairwise different and $J_l \subseteq \mathcal{A}_{\lambda_l}(\mathbf{d})$, $l \in [0, p]$. If \mathcal{C} is an irreducible component of $\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l, J_l}}(\mathbf{d})$, then there exists $M \in \mathcal{C}$ such that $\partial c_{\lambda_0, J_0}(M), \dots, \partial c_{\lambda_p, J_p}(M)$ are linearly independent.*

Proof. We know from Corollary 5.4 that $\dim \mathcal{C} = a_{\Delta}(\mathbf{d}) - p - 1$ and $\mathcal{C} = \overline{\dim(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$ for $\mathbf{d}' \in \mathbf{P} + \mathbf{R}$ and $\mathbf{d}'' \in \mathbf{Q}$. Since $q_{\Delta}(\mathbf{d}'') = 1$, the claim follows from Corollary 5.6. \square

Corollary 6.2. *If $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ are pairwise different, then*

$$\left\{ f \in k[\text{rep}_{\Delta}(\mathbf{d})] : f(M) = 0 \text{ for each } M \in \bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d}) \right\} \\ = (c_{\lambda_0}, \dots, c_{\lambda_p}).$$

Proof. We know from Proposition 3.2(1) that $\text{rep}_{\Delta}(\mathbf{d})$ is a complete intersection. Moreover, the previous lemma implies that for each irreducible component \mathcal{C} of $\bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$ there exists $M \in \mathcal{C}$ such that $\partial c_{\lambda_0}(M), \dots, \partial c_{\lambda_p}(M)$ are linearly independent. Consequently, the claim follows from Propositions 3.1(2). \square

Proposition 6.3. *Let $\lambda_0, \dots, \lambda_p \in \mathbb{X}$ be pairwise different. If $M, N \in \mathcal{R}(\mathbf{d})$ and there exists $\mu \in k$ such that $c_{\lambda_l}(M) = \mu c_{\lambda_l}(N)$ for each $l \in [0, p]$, then M and N are S -equivalent.*

Proof. Lemma 5.1 implies that $c_{\lambda_l}(M) \neq 0$ for some $l \in [0, p]$. Without loss of generality we may assume that $c_{\lambda_0}(M) \neq 0$. Then $\mu \neq 0$ and $c_{\lambda_0}(N) \neq 0$. For $l \in [0, p]$ we put $\mu_l := \frac{c_{\lambda_l}(M)}{c_{\lambda_0}(M)}$. Observe that $c_{\lambda_l}(N) = \mu_l c_{\lambda_0}(N)$ for each $l \in [0, p]$.

Fix $\lambda \in \mathbb{X}$, and put $\mu' := \frac{c_{\lambda}(M)}{c_{\lambda_0}(M)}$ and $\mu'' := \frac{c_{\lambda}(N)}{c_{\lambda_0}(N)}$. We know from Lemma 5.1 that $c_{\lambda}(V) = 0$ for each $V \in \bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l}}(\mathbf{d})$, hence

Corollary 6.2 implies that there exist $f_0, \dots, f_p \in k[\text{rep}_\Delta(\mathbf{d})]$ such that $c_\lambda = \sum_{l \in [0, p]} f_l c_{\lambda_l}$. Put $f := \sum_{l \in [0, p]} \mu_l f_l$. Then

$$\begin{aligned} c_\lambda(g \cdot M) &= \sum_{l \in [0, p]} f_l(g \cdot M) c_{\lambda_l}(g \cdot M) \\ &= \sum_{l \in [0, p]} \mu_l f_l(g \cdot M) c_{\lambda_0}(g \cdot M) = f(g \cdot M) \cdot c_{\lambda_0}(g \cdot M) \end{aligned}$$

for each $g \in \text{GL}(\mathbf{d})$. Recall that c_λ and c_{λ_0} are semi-invariants of the same weight, hence $f(g \cdot M) = \frac{c_\lambda(M)}{c_{\lambda_0}(M)} = \mu'$ for each $g \in \text{GL}(\mathbf{d})$. Similarly, $f(g \cdot N) = \mu''$ for each $g \in \text{GL}(\mathbf{d})$. Since $\overline{\mathcal{O}(M)} \cap \overline{\mathcal{O}(N)} \neq \emptyset$ ($S^{\mathbf{d}} \in \overline{\mathcal{O}(M)} \cap \overline{\mathcal{O}(N)}$), $\mu' = \mu''$. Consequently,

$$c_\lambda(M) = \mu' c_{\lambda_0}(M) = \mu'' \mu c_{\lambda_0}(N) = \mu c_\lambda(N),$$

and the claim follows from Corollary 4.5. \square

Proposition 6.4. *If $\mathcal{O}(M) \subseteq \text{rep}_\Delta(\mathbf{d})$ is maximal, then there exist $\lambda_0, \dots, \lambda_p \in \mathbb{X}$, $i_0 \in \mathcal{A}_{\lambda_0}(\mathbf{d})$, \dots , $i_p \in \mathcal{A}_{\lambda_p}(\mathbf{d})$, and $\mu_1, \dots, \mu_p \in k$, such that*

$$\begin{aligned} \{f \in k[\text{rep}_\Delta(\mathbf{d})] : f(N) = 0 \text{ for each } N \in \overline{\mathcal{O}(M)}\} \\ = (c_{\lambda_1, i_1} - \mu_1 c_{\lambda_0, i_0}, \dots, c_{\lambda_p, i_p} - \mu_p c_{\lambda_0, i_0}). \end{aligned}$$

In particular, $\overline{\mathcal{O}(M)}$ is a complete intersection of dimension $a_\Delta(\mathbf{d}) - p$.

Proof. First, let $(\lambda_1, i_1), \dots, (\lambda_q, i_q)$ be the pairwise different elements of $\hat{\mathbb{X}}(M)$. We put $\mu_l := 0$ for $l \in [1, q]$. Next, we choose pairwise different $\lambda_0, \lambda_{q+1}, \dots, \lambda_p \in \mathbb{X} \setminus (\mathbb{X}_0 \cup \mathbb{X}(M))$. Finally, we put $i_0 := 0$, and $i_l := 0$ and $\mu_l := \frac{c_{\lambda_l}(M)}{c_{\lambda_0}(M)}$ for $l \in [q+1, p]$.

Let

$$\mathcal{V} := \{N \in \text{rep}_\Delta(\mathbf{d}) : c_{\lambda_l, i_l}(N) - \mu_l c_{\lambda_0, i_0}(N) = 0 \text{ for each } l \in [1, p]\}$$

and $\mathcal{V}' := \bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l, i_l}}(\mathbf{d})$. Obviously $\mathcal{V}' \subseteq \mathcal{V}$. Moreover, every irreducible component of \mathcal{V}' has dimension $a_\Delta(\mathbf{d}) - p - 1$ by Corollary 5.4, hence Krull's Principal Ideal Theorem implies that every irreducible component of \mathcal{V} has dimension $a_\Delta(\mathbf{d}) - p$. In particular, Corollary 3.4 implies that $\mathcal{R}(\mathbf{d}) \cap \mathcal{C}$ is a non-empty open subset of \mathcal{C} for each irreducible component \mathcal{C} of \mathcal{V} . Note that $c_{\lambda_l}(R) = \frac{c_{\lambda_0}(R)}{c_{\lambda_0}(M)} c_{\lambda_l}(M)$ for any $l \in [0, p]$ and $R \in \mathcal{R}(\mathbf{d}) \cap \mathcal{V}$, thus Proposition 6.3 implies that R is S-equivalent to M for each $R \in \mathcal{V} \cap \mathcal{R}(\mathbf{d})$. Consequently, there are only finitely many orbits in $\mathcal{R}(\mathbf{d}) \cap \mathcal{V}$. This implies that every irreducible component of \mathcal{V} is of the form $\overline{\mathcal{O}(R)}$ for some $R \in \mathcal{R}(\mathbf{d})$. Fix $R \in \mathcal{R}(\mathbf{d})$ such that $\overline{\mathcal{O}(R)}$ is an irreducible component of \mathcal{V} . Since $\dim \mathcal{O}(R) = a_\Delta(\mathbf{d}) - p$, $\mathcal{O}(R)$ is maximal in $\text{rep}_\Delta(R)$. Moreover, R and M are S-equivalent and $\hat{\mathbb{X}}(M) \subseteq \hat{\mathbb{X}}(R)$, hence $\mathcal{O}(R) = \mathcal{O}(M)$. Consequently, $\mathcal{V} = \overline{\mathcal{O}(M)}$.

Lemma 6.1 implies that there exists $N \in \mathcal{V}$ such that $\partial c_{\lambda_0, i_0}(N), \dots, \partial c_{\lambda_p, i_p}(N)$ are linearly independent. Consequently, there exists $N \in \mathcal{V}$ such that $\partial c_{\lambda_1, i_1}(N) - \mu_1 \partial c_{\lambda_0, i_0}(N), \dots, \partial c_{\lambda_p, i_p}(N) - \mu_p \partial c_{\lambda_0, i_0}(N)$ are linearly independent. Since $\text{rep}_{\Delta}(\mathbf{d})$ is a complete intersection by Proposition 3.2(1), the claim follows from Proposition 3.1(2). \square

Proposition 6.5. *If $\mathcal{O}(M) \subseteq \text{rep}_{\Delta}(\mathbf{d})$ is maximal, then the variety $\overline{\mathcal{O}(M)}$ is normal.*

Proof. We know from Proposition 6.4 that there exist $\lambda_0, \dots, \lambda_p \in \mathbb{X}$, $i_0 \in \mathcal{A}_{\lambda_0}(\mathbf{d}), \dots, i_p \in \mathcal{A}_{\lambda_p}(\mathbf{d})$, and $\mu_1, \dots, \mu_p \in k$, such that

$$\begin{aligned} \{f \in k[\text{rep}_{\Delta}(\mathbf{d})] : f(N) = 0 \text{ for each } N \in \overline{\mathcal{O}(M)}\} \\ = (c_{\lambda_l, i_l} - \mu_l c_{\lambda_0, i_0} : l \in [1, p]). \end{aligned}$$

Let

$$\mathcal{U} := \{N \in \overline{\mathcal{O}(M)} : \dim_k T_N \overline{\mathcal{O}(M)} = \dim \mathcal{O}(M)\}.$$

Equivalently, \mathcal{U} is the set of all $N \in \overline{\mathcal{O}(M)}$ such that $\partial c_{\lambda_1, i_1}(N) - \mu_1 \partial c_{\lambda_0, i_0}(N), \dots, \partial c_{\lambda_p, i_p}(N) - \mu_p \partial c_{\lambda_0, i_0}(N)$ are linearly independent.

By general theory $\mathcal{O}(M) \subseteq \mathcal{U}$, hence $\overline{\mathcal{O}(M)} \setminus \mathcal{U} \subseteq \mathcal{V}' \cup \mathcal{V}''$, where $\mathcal{V}' := \bigcap_{l \in [0, p]} \mathcal{H}^{V_{\lambda_l, i_l}}(\mathbf{d})$ and $\mathcal{V}'' := (\overline{\mathcal{O}(M)} \setminus \mathcal{O}(M)) \cap \mathcal{R}(\mathbf{d})$. Lemma 6.1 says that for each irreducible component \mathcal{C} of \mathcal{V}' there exists $N \in \mathcal{C}$ such that $\partial c_{\lambda_0, i_0}(N), \dots, \partial c_{\lambda_p, i_p}(N)$ are linearly independent. In particular, $\mathcal{U} \cap \mathcal{C} \neq \emptyset$ for each irreducible component \mathcal{C} of \mathcal{V}' , thus $\dim(\mathcal{V}' \setminus \mathcal{U}) < \dim \mathcal{V}' = a_{\Delta}(\mathbf{d}) - p - 1 = \dim \mathcal{O}(M) - 1$. On the other hand, if $R \in \mathcal{V}''$, then R is S-equivalent to M by Proposition 6.3, hence \mathcal{V}'' is a union of finitely many orbits. Moreover, [41, Theorem 1.1] implies that $R \in \mathcal{U}$ for each $R \in \mathcal{V}''$ such that $\dim \mathcal{O}(R) = \dim \mathcal{O}(M) - 1$. Concluding, we obtain that $\dim(\overline{\mathcal{O}(M)} \setminus \mathcal{U}) < \dim \mathcal{O}(M) - 1$. Since $\overline{\mathcal{O}(M)}$ is a complete intersection by Proposition 6.4, the claim follows from Proposition 3.1(1). \square

7. SINGULAR DIMENSION VECTOR

Throughout this section we fix a sincere separating exact subcategory \mathcal{R} of $\text{ind } \Delta$ for a tame bound quiver Δ and use freely notation introduced in Section 2. We also fix singular $\mathbf{d} \in \mathbf{R}$. Proposition 2.3(1) implies that $\mathbf{d} = \mathbf{h}$ and Δ is of type $(2, 2, 2, 2)$. Let $\mathcal{O}(M) \subseteq \text{rep}_{\Delta}(\mathbf{h})$ be maximal. It follows from [7, Proposition 5] that $M \simeq R_{\lambda, i}^{(r_{\lambda})}$ for some $\lambda \in \mathbb{X}$ and $i \in [0, r_{\lambda} - 1]$. We prove that $\overline{\mathcal{O}(M)}$ is normal if and only if $r_{\lambda} = 2$. Note that $\hat{\mathbb{X}}(M) = \{(\lambda, j)\}$, where $j := (i - 1) \bmod r_{\lambda}$. Moreover, $V_{\lambda, j} = R_{\lambda, j}$.

Proposition 7.1. *We have*

$$\{f \in k[\text{rep}_{\Delta}(\mathbf{h})] : f(N) = 0 \text{ for each } N \in \overline{\mathcal{O}(M)}\} = (c_{\lambda, j}).$$

In particular, $\overline{\mathcal{O}(M)}$ is a complete intersection of dimension $a_{\Delta}(\mathbf{h}) - 1$.

Proof. We know from Proposition 3.2(1) that $\text{rep}_\Delta(\mathbf{h})$ is an irreducible variety of dimension $a_\Delta(\mathbf{h})$, hence Krull's Principal Ideal Theorem implies that every irreducible component of $\mathcal{H}^{V_{\lambda,j}}(\mathbf{h})$ has dimension $a_\Delta(\mathbf{h}) - 1$. Observe that $\mathcal{R}(\mathbf{h}) \cap \mathcal{H}^{V_{\lambda,j}}(\mathbf{h})$ is a union of finitely many orbits. Since $\dim(\mathcal{H}^{V_{\lambda,j}}(\mathbf{h}) \setminus \mathcal{R}(\mathbf{h})) \leq a_\Delta(\mathbf{h}) - 2$ by Corollary 3.4, this implies that every irreducible component of \mathcal{V} is of the form $\overline{\mathcal{O}(R)}$ for a maximal orbit $\mathcal{O}(R)$ in $\text{rep}_\Delta(\mathbf{h})$. However, [7, Proposition 5] implies that $\mathcal{O}(M)$ is a unique maximal orbit in $\text{rep}_\Delta(\mathbf{h})$ which is contained in $\mathcal{H}^{V_{\lambda,j}}(\mathbf{h})$, hence $\mathcal{H}^{V_{\lambda,j}}(\mathbf{h}) = \overline{\mathcal{O}(M)}$.

We know that $\dim_k \text{Ext}_\Delta^1(M, M) = 1$ and the non-split exact sequences in $\text{Ext}_\Delta^1(M, M)$ are of the form $\xi : 0 \rightarrow M \rightarrow N \rightarrow M \rightarrow 0$ with $N \simeq R_{\lambda,i}^{(2r_\lambda)}$. In particular, $\dim_k \text{Hom}_\Delta(V_{\lambda,j}, N) = 1$. Consequently, the sequence $0 \rightarrow M \rightarrow M \oplus M \rightarrow M \rightarrow 0$ is the only $V_{\lambda,j}$ -exact sequence in $\text{Ext}_\Delta^1(M, M)$. Propositions 4.6(1) and 3.2(4) imply that $\partial c^{V_{\lambda,j}}(M)$ is non-zero. Since $\text{rep}_\Delta(\mathbf{h})$ is a complete intersection by Proposition 3.2(1), the claim follows from Proposition 3.1(2). \square

Proposition 7.2. *Let*

$$\mathcal{U} := \{N \in \overline{\mathcal{O}(M)} : \dim_k T_N \overline{\mathcal{O}(M)} = \dim \mathcal{O}(M)\}.$$

- (1) *If $r_\lambda = 1$, then $\dim \overline{\mathcal{O}(M)} \setminus \mathcal{U} = \dim \mathcal{O}(M) - 1$. In particular, $\overline{\mathcal{O}(M)}$ is not normal.*
- (2) *If $r_\lambda = 2$, then $\dim \overline{\mathcal{O}(M)} \setminus \mathcal{U} < \dim \mathcal{O}(M) - 1$. In particular, $\overline{\mathcal{O}(M)}$ is normal.*

Proof. Fix $\lambda' \in \mathbb{X} \setminus (\mathbb{X}_0 \cup \{\lambda\})$. Lemma 5.1 implies that $\text{rep}_\Delta(\mathbf{h}) \setminus \mathcal{R}(\mathbf{h}) = \mathcal{H}^{V_\lambda}(\mathbf{h}) \cap \mathcal{H}^{V_{\lambda'}}(\mathbf{h})$. By general theory $\mathcal{O}(M) \subseteq \mathcal{U}$, hence $\overline{\mathcal{O}(M)} \setminus \mathcal{U} \subseteq \mathcal{V}' \cup \mathcal{V}''$, where $\mathcal{V}' := (\overline{\mathcal{O}(M)} \setminus \mathcal{O}(M)) \cap \mathcal{R}(\mathbf{h})$ and $\mathcal{V}'' := \mathcal{H}^{V_{\lambda,j}}(\mathbf{h}) \cap \mathcal{H}^{V_{\lambda'}}(\mathbf{h})$. We know that \mathcal{V}' is a union of finitely many orbits. Moreover, [41, Theorem 1.1] implies that $R \in \mathcal{U}$ for each $R \in \mathcal{V}'$ such that $\dim \mathcal{O}(R) = \dim \mathcal{O}(M) - 1$. Consequently, $\dim(\mathcal{V}' \setminus \mathcal{U}) < \dim \mathcal{V}' \leq \dim \mathcal{O}(M) - 1$.

Now let \mathcal{C} be an irreducible component of \mathcal{V}'' . Corollary 5.4 implies that $\dim \mathcal{C} = a_\Delta(\mathbf{h}) - 2$ and there exist $\mathbf{d}' \in \mathbf{P}$ and $\mathbf{d}'' \in \mathbf{Q}$ such that $\mathcal{C} = \overline{(\mathcal{P} \cup \mathcal{R})(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')}$. Moreover, Corollary 3.4 implies that either $q_\Delta(\mathbf{d}'') = 1$ or $q_\Delta(\mathbf{d}'') = 0$. If $q_\Delta(\mathbf{d}'') = 1$, then Corollary 5.6 implies that $\mathcal{U} \cap \mathcal{C} \neq \emptyset$.

Assume that $q_\Delta(\mathbf{d}'') = 0$ (according to Proposition 2.3(2) this case appears since \mathbf{d}'' is singular). Then $\langle \mathbf{h}, \mathbf{d}'' \rangle_\Delta = 2$ by Corollary 3.4. If $r_\lambda = 2$, then $\langle \dim V_{\lambda,j}, \mathbf{d}'' \rangle_\Delta = 1$. Indeed, we know from Proposition 5.5 that $\langle \dim V_{\lambda,j}, \mathbf{d}'' \rangle_\Delta > 0$. On the other hand, Proposition 2.1(4) implies that $\langle \dim V_{\lambda,j}, \mathbf{d}'' \rangle_\Delta = \langle \mathbf{h}, \mathbf{d}'' \rangle_\Delta - \langle \mathbf{e}_{\lambda,i}, \mathbf{d}'' \rangle_\Delta \leq 2 - 1 = 1$. Consequently, Proposition 5.5 implies that $\mathcal{U} \cap \mathcal{C} \neq \emptyset$ in this case. On the other hand, if $r_\lambda = 1$, then $\dim_k \text{Hom}_\Delta(V_{\lambda,j}, N) \geq$

$\langle \mathbf{h}, \mathbf{d}'' \rangle_{\Delta} = 2$ for each $N \in \mathcal{C}$. Thus, in this case $\mathcal{U} \cap \mathcal{C} = \emptyset$ by Proposition 4.6(2).

Concluding, $\dim(\overline{\mathcal{O}(M)} \setminus \mathcal{U}) < \dim \mathcal{O}(M) - 1$ if and only if $r_{\lambda} = 2$. Since we know from Proposition 7.1 that $\overline{\mathcal{O}(M)}$ is a complete intersection, the claims about (non-)normality of $\overline{\mathcal{O}(M)}$ follow immediately from Proposition 3.1(1). \square

We finish this section with a remark about relationship between the degenerations and the hom-order. Let Δ' be a bound quiver and \mathbf{d}_0 a dimension vector. If $U, V \in \text{rep}_{\Delta'}(\mathbf{d}_0)$, then we say that V is a degeneration of U (and write $U \leq_{\text{deg}} V$) if $\mathcal{O}(V) \subseteq \overline{\mathcal{O}(U)}$. Similarly, we write $U \leq_{\text{hom}} V$ if $\dim_k \text{Hom}_{\Delta'}(X, U) \leq \dim_k \text{Hom}_{\Delta'}(X, V)$ for each $X \in \text{rep } \Delta'$ (equivalently, $\dim_k \text{Hom}_{\Delta'}(U, X) \leq \dim_k \text{Hom}_{\Delta'}(V, X)$ for each $X \in \text{rep } \Delta'$). Both \leq_{deg} and \leq_{hom} induce partial orders in the set of the isomorphism classes of the representations of Δ' . It is also known that \leq_{deg} implies \leq_{hom} . The reverse implication is not true in general, however \leq_{hom} implies \leq_{deg} if either Δ' is of finite representation type [38] or $\text{gl.dim } \Delta' = 1$ and Δ' is of tame representation type [13] (i.e. $R = \emptyset$ and Δ' is an Euclidean quiver). We present an example showing that \leq_{hom} does not imply \leq_{deg} for the tame concealed canonical algebras in general.

We return to the setup of this section and assume that $r_{\lambda} = 2$. Let $R := R_{\lambda,0} \oplus R_{\lambda,1}$. Moreover, we fix $\mathbf{d}'' \in \mathbf{Q}$ such that $q_{\Delta}(\mathbf{d}'') = 0$, $\langle \mathbf{h}, \mathbf{d}'' \rangle_{\Delta} = 2$ and $\mathbf{d}' \in \mathbf{P}$, where $\mathbf{d}' := \mathbf{h} - \mathbf{d}''$. If $N \in \mathcal{P}(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$, then

$$\dim_k \text{Hom}_{\Delta}(R_{\lambda',i'}, M) \leq 1 \leq \dim_k \text{Hom}_{\Delta}(R_{\lambda',i'}, N)$$

for any $\lambda' \in \mathbb{X}$ and $i' \in [0, r_{\lambda'} - 1]$. By adapting [14, Corollary 4.2] to the considered situation, we get that $R \leq_{\text{hom}} N$ for each $N \in \mathcal{P}(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$. On the other hand,

$$\dim \mathcal{O}(R) = a_{\Delta}(\mathbf{d}) - 2 = \dim \mathcal{P}(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}''),$$

hence $\dim \mathcal{P}(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'') \not\subseteq \overline{\mathcal{O}(R)}$, i.e. there exists $N \in \dim \mathcal{P}(\mathbf{d}') \oplus \mathcal{Q}(\mathbf{d}'')$ such that $R \not\leq_{\text{deg}} N$.

8. PROOF OF THEOREM 4

Let M be a periodic representation of a tame concealed-canonical quiver Δ such that $\mathcal{O}(M)$ is maximal.

If $\text{Ext}_{\Delta}^1(M, M) = 0$, then $\overline{\mathcal{O}(M)} = \text{rep}_{\Delta}(\mathbf{d})$ by Proposition 3.2(2). Consequently, $\overline{\mathcal{O}(M)}$ is a normal complete intersection by Proposition 3.2(1). Observe, that $\mathbf{dim } M$ is not singular in this case.

Now assume $\text{Ext}_{\Delta}^1(M, M) \neq 0$. Using Proposition 3.2(3) we may assume that $M \in \text{add } \mathcal{R}$ for a sincere separating exact subcategory \mathcal{R} of $\text{ind } \Delta$. Proposition 3.2(4) implies that $\overline{\mathcal{O}(M)} \neq \text{rep}_{\Delta}(\mathbf{d})$. Consequently, $p^M \neq 0$ (since $\dim \mathcal{O}(M) = \dim \text{rep}_{\Delta}(\mathbf{d}) - p^M$) and the claim follows from Propositions 6.4 and 6.5.

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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, NICOLAUS COPERNICUS UNIVERSITY, UL. CHOPINA 12/18, 87-100 TORUŃ, POLAND

E-mail address: `gregbob@mat.umk.pl`